

Optimal stopping for partially observed piecewise-deterministic Markov processes*

Adrien Brandejsky Benoîte de Saporta François Dufour

Abstract

This paper deals with the optimal stopping problem under partial observation for piecewise-deterministic Markov processes. We first obtain a recursive formulation of the optimal filter process and derive the dynamic programming equation of the partially observed optimal stopping problem. Then, we propose a numerical method, based on the quantization of the discrete-time filter process and the inter-jump times, to approximate the value function and to compute an actual ϵ -optimal stopping time. We prove the convergence of the algorithms and bound the rates of convergence.

Keywords: optimal stopping, partial observation, filtering, piecewise deterministic Markov processes, quantization, numerical method
60G40, 60J25, 93E20, 93E25, 93E10, 60K10

1 Introduction

The aim of this paper is to investigate an optimal stopping problem under partial observation for piecewise-deterministic Markov processes (PDMP) both from the theoretical and numerical points of view. PDMP's have been introduced by Davis [1] as a general class of stochastic models. They form a family of Markov processes involving deterministic motion punctuated by random jumps. The motion depends on three local characteristics, the flow Φ , the jump rate λ and the transition measure Q , which selects the post-jump location. Starting from the point x , the motion of the process $(X_t)_{t \geq 0}$ follows the flow $\Phi(x, t)$ until the first jump time T_1 , which occurs either spontaneously in a Poisson-like fashion with rate $\lambda(\Phi(x, t))$ or when the flow hits the boundary of the state space. In either case, the location of the process at T_1 is selected by the transition measure $Q(\Phi(x, T_1), \cdot)$ and the motion restarts from this new point X_{T_1} . We define similarly the time until the next jump, as well as the next post-jump location and so on. One important property of a PDMP, relevant for the approach developed in this paper, is that its distribution is completely characterized by the embedded discrete time Markov chain $(Z_n, S_n)_{n \in \mathbb{N}}$ where Z_n is the n -th post-jump location and S_n is the n -th inter-jump time. A suitable choice of the state

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space and local characteristics provides stochastic models covering a great number of problems of operations research, see [1, section 33].

In this paper, we consider an optimal stopping problem for a partially observed PDMP $(X_t)_{t \geq 0}$. Roughly speaking, the observation process $(Y_t)_{t \geq 0}$ is a point process defined through the embedded discrete time Markov chain $(Z_n, S_n)_{n \in \mathbb{N}}$. The inter-arrival times are given by $(S_n)_{n \in \mathbb{N}}$ and the marks by a noisy function of $(Z_n)_{n \in \mathbb{N}}$. For a given reward function g and a computation horizon $N \in \mathbb{N}$, we study the following optimal stopping problem

$$\sup_{\sigma \leq T_N} \mathbf{E}[g(X_\sigma)],$$

where T_N is the N -th jump time of the PDMP $(X_t)_{t \geq 0}$, σ is a stopping time with respect to the natural filtration $\mathfrak{F}^Y = (\mathfrak{F}_t^Y)_{t \geq 0}$ generated by the observations $(Y_t)_{t \geq 0}$.

A general methodology to solve such a problem is to split it into two sub-problems. The first one consists in deriving the filter process given by the conditional expectation of X_t with respect to the observed information \mathfrak{F}_t^Y . Its main objective is to transform the initial problem into a completely observed optimal stopping problem where the new state variable is the filter process. The second step consists in solving this reformulated problem, the new difficulty being its infinite dimension. Indeed, the filter process takes values in a set of probability measures.

Our work is inspired by [2] which deals with an optimal stopping problem under partial observation for a Markov chain with finite state space. The authors study the optimal filtering and convert their original problem into a standard optimal stopping problem for a continuous state space Markov chain. Then they propose a discretization method based on a quantization technique to approximate the value function. However, their method cannot be directly applied to our problem for the following main reasons related to the specificities of PDMPs.

Firstly, PDMPs are continuous time processes. Then, it appears natural to work with the embedded Markov chain $(Z_n, S_n)_{n \in \mathbb{N}}$. In addition, we assume that $(Z_n)_{n \in \mathbb{N}}$ takes finitely many values. However, an important difficulty is that the structure of stopping time remains intrinsically continuous. Consequently, our problem cannot be converted into a fully discrete time problem.

Secondly, the distribution of a PDMP combines both absolutely continuous and singular components. This is due to the existence of forced jumps when the process hits the boundary of the state space. As a consequence the derivation of the filter process is not straightforward. In particular, the absolute continuity hypothesis **(H)** of [2] does not hold.

Thirdly, in our context the reformulated optimization problem is not standard, unlike in [2]. Indeed, although we obtain a reformulation similar to an optimal stopping problem for a fully observed PDMP, it involves the Markov chain $(\Pi_n, S_n)_{n \in \mathbb{N}}$ that is not the embedded Markov chain of some PDMP. Therefore, a new derivation of dynamic programming equations is required as we cannot use the results of [4]. In particular, one needs to derive fine properties of the structure of the $(\mathfrak{F}_t^Y)_{t \geq 0}$ -stopping times. Moreover, we construct an ϵ -optimal stopping time.

Finally, a natural way to proceed with the numerical approximation is then to follow the ideas developed in [2, 5] namely to replace the filter Π_n and the inter-jump time S_n by some finite state space approximations in the dynamic programming

equation. However, a noticeable difference from [5] lies in the fact that the dynamic programming operators therein were Lipschitz continuous whereas our new operators are only Lipschitz continuous between some points of discontinuity. We overcome this drawback by splitting the operators into their restrictions onto their continuity sets. This way, we obtain not only an approximation of the value function of the optimal stopping problem but also an ϵ -optimal stopping time with respect to the filtration $(\mathfrak{F}_t^Y)_{t \geq 0}$ that can be computed in practice.

Our approximation procedure for random variables is based on quantization. There exists an extensive literature on this method. The interested reader may for instance consult [7, 8] and the references within. The quantization of a random variable X consists in finding a finite grid such that the projection \widehat{X} of X on this grid minimizes some L^p norm of the difference $X - \widehat{X}$. Roughly speaking, such a grid will have more points in the areas of high density of X . As explained for instance in [8, section 3], under some Lipschitz-continuity conditions, bounds for the rate of convergence of functionals of the quantized process towards the original process are available, which makes this technique especially appealing. Quantization methods have been developed recently in numerical probability or optimal stochastic control with applications in finance, see e.g. [8, 9, 10].

The paper is organized as follows. Section 2 introduces the notation, recalls the definition of a PDMP, presents our assumptions and defines the optimal stopping problem we are interested in, especially the observation process. The recursive formulation of the filter process is derived in Section 3. In Section 4, we reduce our partially observed problem for the PDMP $(X_t)_{t \geq 0}$ to a completely observed one involving the process $(\Pi_n, S_n)_{n \in \mathbb{N}}$ for which we provide the dynamic programming equation and construct a family of ϵ -optimal stopping times. Then, our numerical methods to compute the value function and an ϵ -optimal stopping time are presented in Section 5 where we also prove the convergence of our algorithms after having recalled the main features of quantization. Finally, an academic example is discussed in Section 6 while technical results are postponed to the Appendices.

2 Definition and notation

In this first section, let us define a piecewise-deterministic Markov process (PDMP) and introduce some general assumptions. For any metric space E , we denote $\mathcal{B}(E)$ its Borel σ -field, $B(E)$ the set of real-valued, bounded and measurable functions defined on E and $BL(E)$ the subset of functions of $B(E)$ that are Lipschitz continuous. For $a, b \in \mathbb{R}$, denote $a \wedge b = \min(a, b)$ and $a \vee b = \max(a, b)$.

2.1 Definition of a Piecewise-Deterministic Markov Process

Let E be an open subset of \mathbb{R}^d . Let ∂E be its boundary and \overline{E} its closure and for any subset A of E , A^c denotes its complement. A PDMP is defined by its local characteristics (Φ, λ, Q) .

- The flow $\Phi : \mathbb{R}^d \times \mathbb{R}^+ \rightarrow \mathbb{R}^d$ is continuous. For all $t \in \mathbb{R}^+$, $\Phi(\cdot, t)$ is an homeomorphism and $t \rightarrow \Phi(\cdot, t)$ is a semi-group: for all $x \in \mathbb{R}^d$, $\Phi(x, t +$

$s) = \Phi(\Phi(x, s), t)$. For all $x \in E$, define the deterministic exit time from E : $t^*(x) = \inf\{t > 0 \text{ such that } \Phi(x, t) \in \partial E\}$. We use here and throughout the convention $\inf \emptyset = +\infty$.

- The jump rate $\lambda : \bar{E} \rightarrow \mathbb{R}^+$ is measurable and satisfies:

$$\forall x \in E, \exists \epsilon > 0 \text{ such that } \int_0^\epsilon \lambda(\Phi(x, t)) dt < +\infty.$$

- Finally, Q is a Markov kernel on $(\bar{E}, \mathcal{B}(\bar{E}))$ which satisfies:

$$\forall x \in \bar{E}, Q(x, E \setminus \{x\}) = 1.$$

From these characteristics, it can be shown [1] that there exists a filtered probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t \in \mathbb{R}^+}, (\mathbf{P}_x)_{x \in E})$ on which a process $(X_t)_{t \in \mathbb{R}^+}$ is defined. Its motion, starting from a point $x \in E$, may be constructed as follows. Let T_1 be a nonnegative random variable with survival function:

$$\mathbf{P}_x(T_1 > t) = \begin{cases} e^{-\Lambda(x, t)} & \text{if } 0 \leq t < t^*(x), \\ 0 & \text{if } t \geq t^*(x), \end{cases}$$

where for $x \in E$ and $t \in [0, t^*(x)]$, $\Lambda(x, t) = \int_0^t \lambda(\Phi(x, s)) ds$. One then chooses an E -valued random variable Z_1 with distribution $Q(\Phi(x, T_1), \cdot)$. The trajectory of X_t for $t \leq T_1$ is:

$$X_t = \begin{cases} \Phi(x, t) & \text{if } t < T_1, \\ Z_1 & \text{if } t = T_1. \end{cases}$$

Starting from the point $X_{T_1} = Z_1$, one selects in a similar way $S_2 = T_2 - T_1$ the time between T_1 and the next jump time T_2 , as well as Z_2 the next post-jump location and so on. Davis showed [1] that the process so defined is a strong Markov process $(X_t)_{t \geq 0}$ with jump times $(T_n)_{n \in \mathbb{N}}$ ($T_0 = 0$). The process $(Z_n, S_n)_{n \in \mathbb{N}}$ where $Z_n = X_{T_n}$ the n -th post-jump location and $S_n = T_n - T_{n-1}$ ($S_0 = 0$) is the n -th inter-jump time is clearly a discrete-time Markov chain.

2.2 Notation and assumptions

The following non explosion assumption about the jump-times is standard (see for example [1, section 24]).

Assumption 2.1. For all $(x, t) \in E \times \mathbb{R}^+$, $\mathbf{E}_x \left[\sum_k \mathbb{1}_{\{T_k < t\}} \right] < +\infty$.

It implies that $T_k \rightarrow +\infty$ a.s. when $k \rightarrow +\infty$. Moreover, we make the following assumption about the transition kernel Q .

Assumption 2.2. We assume that there exists a finite set $E_0 = \{x_1, \dots, x_q\} \subset E$ such that for all $x \in E$, one has $Q(x, E_0) = 1$.

In other words, for all $n \in \mathbb{N}$, Z_n may only take its values in the finite set E_0 . This assumption ensures that the filter process, defined in the next section, has finite dimension. This is required to derive a tractable numerical method in Section 5. When this assumption does not hold, one may consider a preliminary discretization of the transition kernel to introduce it.

Assumption 2.3. We assume that the function t^* is bounded on E_0 i.e. for all $m \in \{1, \dots, q\}$, we assume that $0 < t^*(x_m) < +\infty$.

Definition 2.4. For all $m \in \{1, \dots, q\}$, denote $t_m^* = t^*(x_m)$ and assume that x_1, \dots, x_q are numbered such that $t_1^* \leq t_2^* \leq \dots \leq t_q^*$. Moreover, let $t_0^* = 0$.

For any function w in $B(E)$, introduce the following notation

$$Qw(x) = \int_E w(y)Q(x, dy) = \sum_{i=1}^q w(x_i)Q(x, x_i), \quad C_w = \sup_{x \in \bar{E}} |w(x)|.$$

For any Lipschitz continuous function w in $BL(E)$, denote $[w]$ its Lipschitz constant

$$[w] = \sup_{x \neq y \in E} \frac{|w(x) - w(y)|}{|x - y|}.$$

Assumption 2.5. The jump jump rate λ is in $B(\bar{E})$ i.e. is bounded by C_λ .

Denote $\mathcal{M}(E_0)$ the set of finite signed measures on E_0 and $\mathcal{M}_1(E_0)$ the subset of probability measures on E_0 . We equip $\mathcal{M}(E_0)$ with the norm $|\cdot|$ given by $|\pi| = \sum_{i=1}^q |\pi^i|$ where π^i denotes $\pi(\{x_i\})$.

2.3 Partially observed optimal stopping problem

We consider from now on a PDMP $(X_t)_{t \geq 0}$ which initial state $X_0 = Z_0$ is a fixed point $x_0 \in E$. We assume that this PDMP is observed through a noise and we now turn to the description of our observation procedure.

For all $n \in \mathbb{N}$, we assume that S_n is perfectly observed but that Z_n is not (except for the initial state Z_0). In some examples, it seems reasonable to consider that the jump times of the process are observed (for instance, if the jumps correspond to changes of environment) and that, when a jump occurs, the actual post-jump location is measured with a noise. The *observation* process of Z_n , denoted Y_n is assumed to be of the following form: $Y_0 = x_0$ (deterministic) and for $n \geq 1$,

$$Y_n = \varphi(Z_n) + W_n, \tag{1}$$

where $\varphi : E_0 \rightarrow \mathbb{R}^d$ and where the *noise* $(W_n)_{n \geq 1}$ is a sequence of \mathbb{R}^d -valued, i.i.d. random variables with bounded density function f_W that are also independent from $(Z_n, S_n)_{n \in \mathbb{N}}$.

In order to define real-valued stopping times adapted to the observation process, we need to consider a continuous time version of the observation process. We therefore define the piecewise-constant process $(Y_t)_{t \geq 0}$ with a slight abuse of notation¹ as

$$Y_t = \sum_{j=0}^{+\infty} \mathbb{1}_{[T_j, T_{j+1}[}(t) Y_j.$$

Let $\mathfrak{F}^Y = (\mathfrak{F}_t^Y)_{t \geq 0}$ be the filtration generated by $(Y_t)_{t \geq 0}$ (the *observed* filtration) and $\mathfrak{F} = (\mathfrak{F}_t)_{t \geq 0}$ be the filtration generated by $(X_t, Y_t)_{t \geq 0}$ (the *total* filtration). Without changing the notation, we then complete these filtrations with all the \mathbf{P} -null sets. This leads us to the following definition.

¹The quantity Y_n represents the value of the process $(Y_t)_{t \geq 0}$ at time $t = T_n$ and must not be confused with the value of the process at time $t = n$.

Definition 2.6. Denote Σ^Y the set of $(\mathfrak{F}_t^Y)_{t \geq 0}$ -stopping times that are a.s. finite and for $n \in \mathbb{N}$, define

$$\Sigma_n^Y = \left\{ \sigma \in \Sigma^Y \text{ such that } \sigma \leq T_n \text{ a.s.} \right\}.$$

For all $n \in \mathbb{N}$, we define the filter $\Pi_n \in \mathcal{M}_1(E_0)$. The quantity $\Pi_n(\{x_i\})$, denoted by Π_n^i , represents the probability of the event $\{Z_n = x_i\}$ given the information available until time T_n i.e.

$$\forall i \in \{1, \dots, q\}, \quad \Pi_n^i = \mathbf{E}[\mathbb{1}_{\{Z_n = x_i\}} | \mathfrak{F}_{T_n}^Y]. \quad (2)$$

Finally, let $N \in \mathbb{N}$ be the *horizon* and $g \in B(\bar{E})$ the *reward function*, we are interested in the following partially observed optimal stopping problem

$$v(\pi) = \sup_{\sigma \in \Sigma_N^Y} \mathbf{E} \left[g(X_\sigma) | \Pi_0 = \pi \right], \quad (3)$$

where π is a probability measure in $\mathcal{M}_1(E_0)$. The solution of our problem is then obtained by setting $\pi = \delta_{x_0}$. We will also need the following assumption about the reward function g associated with the optimal stopping problem.

Assumption 2.7. The function g is in $B(\bar{E})$ i.e. bounded by C_g and there exists $[g]_2 \in \mathbb{R}^+$ such that for all $i \in \{1, \dots, q\}$ and $t, u \in [0, t_i^*]$, one has:

$$|g(\Phi(x_i, t)) - g(\Phi(x_i, u))| \leq [g]_2 |t - u|.$$

Now, the aims of this paper are first to explicit the filter process $(\Pi_n)_{n \in \mathbb{N}}$ (Section 3); second to rewrite the partially observed optimal stopping problem (3) as a totally observed one for a suitable Markov chain on $\mathcal{M}_1(E_0) \times \mathbb{R}^+$ (Section 4.3); third to derive a dynamic programming equation and construct a family of ϵ -optimal stopping times (Sections 4.4 and 4.5); and finally to propose a numerical method to compute an approximation of the value function and an ϵ -optimal stopping time (Section 5). As a starting point, we will derive, in the next section, a recursive construction of the optimal filter that is the key point of our approach.

3 Optimal filtering

The goal of this section is to obtain a recursive formulation of the filter Π_n . As far as we know, there is no result concerning the filter process for generic PDMP's. We may however refer to [11] for a recursive formulation of the filter for point processes, that can be seen as a sub-class of PDMP's. For all $n \in \mathbb{N}$, we denote $\mathcal{G}_n = (Y_0, S_0, \dots, Y_n, S_n)$. The continuous-time observation process $(Y_t)_{t \geq 0}$ being a point process in the sense developed in [6], one has $\mathfrak{F}_{T_n}^Y = \sigma(\mathcal{G}_n)$ (see [6, page 58, Theorem T2]). Moreover, $\mathfrak{F}_{T_n} = \sigma(Z_0, \dots, Z_n) \vee \mathfrak{F}_{T_n}^Y$. Concerning the filter Π_n , first notice that, since it is an $\mathfrak{F}_{T_n}^Y$ -measurable random variable, there exists for all $n \in \mathbb{N}$ a measurable function $\pi_n : (\mathbb{R}^d \times \mathbb{R}^+)^{n+1} \rightarrow \mathcal{M}_1(E_0)$ such that $\Pi_n = \pi_n(\mathcal{G}_n)$. As in the case of the Kalman-Bucy filter, the iteration leading from Π_{n-1} to Π_n can be split into two steps : prediction and correction. For all $n \geq 1$, let μ_n^- be

the conditional distribution of (Z_n, S_n) given $\mathfrak{F}_{T_{n-1}}^Y$. Thus, μ_n^- is a transition kernel defined on $(\mathbb{R}^d \times \mathbb{R}^+)^n \times \mathcal{B}(E_0 \times \mathbb{R}^+)$ for all $j \in \{1, \dots, q\}$ and $\gamma_{n-1} \in (\mathbb{R}^d \times \mathbb{R}^+)^n$ by

$$\mu_n^-(\gamma_{n-1}, \{x_j\}, ds) = \mathbf{P}(Z_n = x_j, S_n \in ds | \mathcal{G}_{n-1} = \gamma_{n-1}). \quad (4)$$

Lemma 3.1. *For all $\gamma_{n-1} \in (\mathbb{R}^d \times \mathbb{R}^+)^n$, we have the following equality of probability measures on $E_0 \times \mathbb{R}^d \times \mathbb{R}^+$, for all $j \in \{1, \dots, q\}$,*

$$\mathbf{P}(Z_n = x_j, Y_n \in dy, S_n \in ds | \mathcal{G}_{n-1} = \gamma_{n-1}) = \mu_n^-(\gamma_{n-1}, \{x_j\}, ds) f_W(y - \varphi(x_j)) dy.$$

Proof Set h in $B(E_0 \times \mathbb{R}^d \times \mathbb{R}^+)$, using Eq. (1) that defines Y_n , one has

$$\begin{aligned} & \mathbf{E} [h(Z_n, Y_n, S_n) | \mathcal{G}_{n-1} = \gamma_{n-1}] \\ &= \mathbf{E} [h(Z_n, \varphi(Z_n) + W_n, S_n) | \mathcal{G}_{n-1} = \gamma_{n-1}] \\ &= \sum_{j=1}^q \int h(x_j, \varphi(x_j) + w, s) \mathbf{P}(Z_n = x_j, S_n \in ds, W_n \in dw | \mathcal{G}_{n-1} = \gamma_{n-1}). \end{aligned}$$

Moreover, W_n is independent from $\sigma(Z_n, S_n) \vee \mathfrak{F}_{T_{n-1}}^Y = \sigma(Z_n, S_n, \mathcal{G}_{n-1})$ and admits the density function f_W . One has then

$$\begin{aligned} & \mathbf{E} [h(Z_n, Y_n, S_n) | \mathcal{G}_{n-1} = \gamma_{n-1}] \\ &= \sum_{j=1}^q \int h(x_j, \varphi(x_j) + w, s) \mathbf{P}(Z_n = x_j, S_n \in ds | \mathcal{G}_{n-1} = \gamma_{n-1}) f_W(w) dw \\ &= \sum_{j=1}^q \int h(x_j, y, s) \mathbf{P}(Z_n = x_j, S_n \in ds | \mathcal{G}_{n-1} = \gamma_{n-1}) f_W(y - \varphi(x_j)) dy. \end{aligned}$$

The last equality is obtained by the change of variable $y = \varphi(x_j) + w$ and gives the result. \square

Integrating w.r.t. to the first variable in the previous lemma (i.e. summing w.r.t. x_j) yields the following result.

Lemma 3.2. *For all $\gamma_{n-1} \in (\mathbb{R}^d \times \mathbb{R}^+)^n$, we have the following equality of probability measures on $\mathbb{R}^d \times \mathbb{R}^+$,*

$$\mathbf{P}(Y_n \in dy, S_n \in ds | \mathcal{G}_{n-1} = \gamma_{n-1}) = \left[\sum_{j=1}^q \mu_n^-(\gamma_{n-1}, \{x_j\}, ds) f_W(y - \varphi(x_j)) \right] dy.$$

Lemma 3.3. *For all $n \geq 1$, $\gamma_{n-1} \in (\mathbb{R}^d \times \mathbb{R}^+)^n$ and $j \in \{1, \dots, q\}$, the distribution μ_n^- , defined by Eq. (4), satisfies*

$$\begin{aligned} & \mu_n^-(\gamma_{n-1}, \{x_j\}, ds) \\ &= \sum_{m=0}^{q-1} \mathbb{1}_{\{s \in [t_m^*, t_{m+1}^*)\}} \left(\sum_{i=m+1}^q \pi_{n-1}^i(\gamma_{n-1}) \lambda(\Phi(x_i, s)) e^{-\Lambda(x_i, s)} Q(\Phi(x_i, s), x_j) \right) ds \\ &+ \sum_{m=1}^q \left(\pi_{n-1}^m(\gamma_{n-1}) e^{-\Lambda(x_m, t_m^*)} Q(\Phi(x_m, t_m^*), x_j) \right) \delta_{t_m^*}(ds). \end{aligned}$$

Proof Let h be a function of $B(E_0 \times \mathbb{R}^+)$. Since $\sigma(\mathcal{G}_{n-1}) = \mathfrak{F}_{T_{n-1}}^Y \subset \mathfrak{F}_{T_{n-1}}$, the law of iterated conditional expectations yields

$$\mathbf{E} \left[h(Z_n, S_n) \middle| \mathcal{G}_{n-1} = \gamma_{n-1} \right] = \mathbf{E} \left[\mathbf{E} \left[h(Z_n, S_n) \middle| \mathfrak{F}_{T_{n-1}} \right] \middle| \mathcal{G}_{n-1} = \gamma_{n-1} \right].$$

Besides, $\mathfrak{F}_{T_{n-1}} = \sigma(Z_0, S_0, W_0, \dots, Z_{n-1}, S_{n-1}, W_{n-1})$ so that

$$\mathbf{E} \left[h(Z_n, S_n) \middle| \mathfrak{F}_{T_{n-1}} \right] = \mathbf{E} \left[h(Z_n, S_n) \middle| Z_0, S_0, \dots, Z_{n-1}, S_{n-1} \right],$$

by independence of the sequences $(W_n)_{n \in \mathbb{N}}$ and $(Z_n, S_n)_{n \in \mathbb{N}}$. Now, we apply the Markov property of $(Z_n, S_n)_{n \in \mathbb{N}}$ to obtain

$$\mathbf{E} \left[h(Z_n, S_n) \middle| \mathfrak{F}_{T_{n-1}} \right] = \mathbf{E} \left[h(Z_n, S_n) \middle| Z_{n-1}, S_{n-1} \right],$$

and finally, a well-known special feature of the transition kernel of the underlying Markov chain of a PDMP provides

$$\mathbf{E} \left[h(Z_n, S_n) \middle| \mathfrak{F}_{T_{n-1}} \right] = \mathbf{E} \left[h(Z_n, S_n) \middle| Z_{n-1} \right].$$

Moreover, the transition kernel can be explicitly expressed in terms of the local characteristics of the PDMP, and this yields the next equations

$$\begin{aligned} & \mathbf{E} \left[h(Z_n, S_n) \middle| \mathcal{G}_{n-1} = \gamma_{n-1} \right] \\ &= \mathbf{E} \left[\sum_{i=1}^q \mathbb{1}_{\{Z_{n-1}=x_i\}} \mathbf{E} \left[h(Z_n, S_n) \middle| Z_{n-1} = x_i \right] \middle| \mathcal{G}_{n-1} = \gamma_{n-1} \right] \\ &= \mathbf{E} \left[\sum_{i=1}^q \mathbb{1}_{\{Z_{n-1}=x_i\}} \sum_{j=1}^q \left[\int_{\mathbb{R}^+} h(x_j, s) \lambda(\Phi(x_i, s)) e^{-\Lambda(x_i, s)} \mathbb{1}_{\{s < t_i^*\}} Q(\Phi(x_i, s), x_j) ds \right. \right. \\ & \quad \left. \left. + h(x_j, t_i^*) e^{-\Lambda(x_i, t_i^*)} Q(\Phi(x_i, t_i^*), x_j) \right] \middle| \mathcal{G}_{n-1} = \gamma_{n-1} \right] \\ &= \sum_{j=1}^q \left[\int_{\mathbb{R}^+} h(x_j, s) \sum_{i=1}^q \pi_{n-1}^i(\gamma_{n-1}) \lambda(\Phi(x_i, s)) e^{-\Lambda(x_i, s)} \mathbb{1}_{\{s < t_i^*\}} Q(\Phi(x_i, s), x_j) ds \right. \\ & \quad \left. + \sum_{i=1}^q h(x_j, t_i^*) \pi_{n-1}^i(\gamma_{n-1}) e^{-\Lambda(x_i, t_i^*)} Q(\Phi(x_i, t_i^*), x_j) \right]. \end{aligned}$$

This can be written equivalently as

$$\begin{aligned} & \mathbf{E} \left[h(Z_n, S_n) \middle| \mathcal{G}_{n-1} = \gamma_{n-1} \right] \\ &= \sum_{j=1}^q \left[\sum_{m=0}^{q-1} \left(\int_{t_m^*}^{t_{m+1}^*} h(x_j, s) \sum_{i=m+1}^q \pi_{n-1}^i(\gamma_{n-1}) \lambda(\Phi(x_i, s)) e^{-\Lambda(x_i, s)} Q(\Phi(x_i, s), x_j) \right) ds \right. \\ & \quad \left. + \sum_{i=1}^q h(x_j, t_i^*) \pi_{n-1}^i(\gamma_{n-1}) e^{-\Lambda(x_i, t_i^*)} Q(\Phi(x_i, t_i^*), x_j) \right]. \end{aligned}$$

Hence the result. \square

We now state the main result of this section, namely the recursive formulation of the filter sequence $(\Pi_n)_{n \in \mathbb{N}}$.

Proposition 3.4. Let $\Psi = (\Psi^1, \dots, \Psi^q) : \mathcal{M}_1(E_0) \times \mathbb{R}^d \times \mathbb{R}^+ \rightarrow \mathcal{M}_1(E_0)$ be defined as follows: for all $j \in \{1, \dots, q\}$,

$$\Psi^j(\pi, y, s) = \sum_{m=0}^{q-1} \mathbb{1}_{\{s \in]t_m^*, t_{m+1}^*]\}} \frac{\Psi_m^j(\pi, y, s)}{\bar{\Psi}_m(\pi, y, s)} + \sum_{m=1}^q \mathbb{1}_{\{s=t_m^*\}} \frac{\Psi_m^{*j}(y)}{\bar{\Psi}_m^*(y)},$$

where

$$\begin{aligned} \Psi_m^j(\pi, y, s) &= \sum_{i=m+1}^q \pi^i \lambda(\Phi(x_i, s)) e^{-\Lambda(x_i, s)} Q(\Phi(x_i, s), x_j) f_W(y - \varphi(x_j)), \\ \bar{\Psi}_m(\pi, y, s) &= \sum_{k=1}^q \Psi_m^k(\pi, y, s), \\ \Psi_m^{*j}(y) &= Q(\Phi(x_m, t_m^*), x_j) f_W(y - \varphi(x_j)), \\ \bar{\Psi}_m^*(y) &= \sum_{k=1}^q \Psi_m^{*k}(y). \end{aligned}$$

Then, the filter, defined in Eq. (2), satisfies $\Pi_0^j = \mathbf{P}(Z_0 = x_j)$ and the following recursion: for all $n \geq 1$,

$$\mathbf{P}\text{-a.s.}, \quad \Pi_n = \Psi(\Pi_{n-1}, Y_n, S_n).$$

Proof Fix γ_{n-1} in $(\mathbb{R}^d \times \mathbb{R}^+)^n$. Bayes formula yields for all $j \in \{1, \dots, q\}$,

$$\begin{aligned} \mathbf{P}(Z_n = x_j, Y_n \in dy, S_n \in ds | \mathcal{G}_{n-1} = \gamma_{n-1}) &= \\ \mathbf{P}(Z_n = x_j | \mathcal{G}_n = (\gamma_{n-1}, y, s)) \times \mathbf{P}(Y_n \in dy, S_n \in ds | \mathcal{G}_{n-1} = \gamma_{n-1}). \end{aligned}$$

Lemmas 3.1 and 3.2 yield

$$\begin{aligned} &\mu_n^-(\gamma_{n-1}, \{x_j\}, ds) f_W(y - \varphi(x_j)) dy \\ &= \mathbf{P}(Z_n = x_j | \mathcal{G}_n = (\gamma_{n-1}, y, s)) \left[\sum_{k=1}^q \mu_n^-(\gamma_{n-1}, \{x_k\}, ds) f_W(y - \varphi(x_k)) \right] dy. \end{aligned}$$

With respect to y , one recognizes the equality of two absolutely continuous measures which implies the equality a.e. of the density functions. Thus, one has for almost all $y \in \mathbb{R}^d$ w.r.t. the Lebesgue measure,

$$\begin{aligned} &\mu_n^-(\gamma_{n-1}, \{x_j\}, ds) f_W(y - \varphi(x_j)) \\ &= \mathbf{P}(Z_n = x_j | \mathcal{G}_n = (\gamma_{n-1}, y, s)) \left[\sum_{k=1}^q \mu_n^-(\gamma_{n-1}, \{x_k\}, ds) f_W(y - \varphi(x_k)) \right]. \end{aligned} \tag{5}$$

Eq. (5) states the equality of two measures of the variable $s \in \mathbb{R}^+$ that contain both an absolutely continuous part and some weighted Dirac measures. Denote $g_1(y, s) \nu_1(ds)$ (respectively $g_2(y, s) \nu_2(ds)$) the left-hand (resp. right-hand) side term of the previous equality. Eq. (5) means that for all function $F \in B(\mathbb{R}^+)$ and for almost all $y \in \mathbb{R}^d$ w.r.t. the Lebesgue measure, one has

$$\int F(s) g_1(y, s) \nu_1(ds) = \int F(s) g_2(y, s) \nu_2(ds), \tag{6}$$

Recall that, from Lemma 3.3, the distribution $\mu_n^-(\gamma_{n-1}, \{x_j\}, ds)$ has a density on the interval $]t_m^*; t_{m+1}^*[$, that we will denote $f_m(\gamma_{n-1}, x_j, s)$, given by

$$f_m(\gamma_{n-1}, x_j, s) = \sum_{i=m+1}^q \pi_{n-1}^i(\gamma_{n-1}) \lambda(\Phi(x_i, s)) e^{-\Lambda(x_i, s)} Q(\Phi(x_i, s), x_j).$$

First, take $F(s) = H(s) \mathbb{1}_{\{s \in]t_m^*; t_{m+1}^*[\}}$ in Eq. (6): for all $H \in B(\mathbb{R}^+)$, one has

$$\begin{aligned} & \int_{t_m^*}^{t_{m+1}^*} H(s) f_m(\gamma_{n-1}, x_j, s) f_W(y - \varphi(x_j)) ds \\ &= \int_{t_m^*}^{t_{m+1}^*} H(s) \mathbf{P}(Z_n = x_j | \mathcal{G}_n = (\gamma_{n-1}, y, s)) \sum_{k=1}^q f_m(\gamma_{n-1}, x_k, s) f_W(y - \varphi(x_k)) ds, \end{aligned}$$

and thus on $]t_m^*; t_{m+1}^*[$, almost surely w.r.t. the Lebesgue measure, one has

$$\mathbf{P}(Z_n = x_j | \mathcal{G}_n = (\gamma_{n-1}, y, s)) = \frac{f_m(\gamma_{n-1}, x_j, s) f_W(y - \varphi(x_j))}{\sum_{k=1}^q f_m(\gamma_{n-1}, x_k, s) f_W(y - \varphi(x_k))}.$$

Finally, for $m \in \{1, \dots, q\}$, choosing $F(s) = \mathbb{1}_{\{s=t_m^*\}}$ in Eq. (6) yields the equality of the weights at the point t_m^* thus, using Lemma 3.3,

$$\begin{aligned} & \mathbf{P}(Z_n = x_j | \mathcal{G}_n = (\gamma_{n-1}, y, t_m^*)) \\ &= \frac{\pi_{n-1}^m(\gamma_{n-1}) e^{-\Lambda(x_m, t_m^*)} Q(\Phi(x_m, t_m^*), x_j) f_W(y - \varphi(x_j))}{\sum_{k=1}^q \pi_{n-1}^m(\gamma_{n-1}) e^{-\Lambda(x_m, t_m^*)} Q(\Phi(x_m, t_m^*), x_k) f_W(y - \varphi(x_k))} \\ &= \frac{Q(\Phi(x_m, t_m^*), x_j) f_W(y - \varphi(x_j))}{\sum_{k=1}^q Q(\Phi(x_m, t_m^*), x_k) f_W(y - \varphi(x_k))}. \end{aligned}$$

Thus there exists two measurable sets $N_y \subset \mathbb{R}^d$ and $N_s \subset \mathbb{R}^+ \setminus \{t_1^*, \dots, t_q^*\}$, negligible w.r.t. the Lebesgue measures on \mathbb{R}^d and \mathbb{R} respectively, such that for all $\gamma_{n-1} \in (\mathbb{R}^d \times \mathbb{R}^+)^n$, $y \in \mathbb{R}^d \setminus N_y$, $s \in \mathbb{R}^+ \setminus N_s$, one has

$$\pi_n(\gamma_{n-1}, y, s) = \Psi(\pi_{n-1}(\gamma_{n-1}), y, s). \quad (7)$$

On the one hand, we have $\mathbf{P}(Y_n \in N_y) \leq \sum_{j=1}^q \mathbf{P}(\varphi(x_j) + W_n \in N_y) = 0$ by absolute continuity of the distribution of W_n . On the other hand, $\mathbf{P}(S_n \in N_s) = 0$ because the distribution of S_n is absolutely continuous on $\mathbb{R}^+ \setminus \{t_1^*, \dots, t_q^*\}$ and one has $N_s \cap \{t_1^*, \dots, t_q^*\} = \emptyset$. We therefore conclude from Eq. (7) that \mathbf{P} -a.s., one has $\pi_n(\mathcal{G}_{n-1}, Y_n, S_n) = \Psi(\pi_{n-1}(\mathcal{G}_{n-1}), Y_n, S_n)$. The result follows since \mathbf{P} -a.s., one has $\pi_n(\mathcal{G}_{n-1}, Y_n, S_n) = \Pi_n$ and $\pi_{n-1}(\mathcal{G}_{n-1}) = \Pi_{n-1}$. \square

This proposition will play a crucial part in the sequel. On the one hand, this result will enable us to prove the Markov property of the sequence $(\Pi_n, S_n)_{n \geq 0}$ w.r.t. the observed filtration. On the other hand, the recursive formulation allows for simulation of the process $(\Pi_n)_{n \geq 0}$ which is crucial to obtain numerical approximations. Finally, notice that the specific structure of the PDMP appears in the recursive formulation of the filter which contains both an absolutely continuous part and some weighted points.

4 Dynamic programming

In this section, we derive the dynamic programming equation for the value function of the partially observed optimal stopping problem (3). After a preliminary study of the structure of the stopping times of Σ_N^Y , the first step consists in converting the partially observed optimal stopping problem into an optimal stopping problem under complete observation. Then, we introduce some operators to recursively build a sequence of function $(v_n)_{0 \leq n \leq N}$ that are the value functions of the optimal stopping problems with horizon T_{N-n} . In particular, v_0 is the value function of the optimal stopping problem (3) we are interested in. We also provide a family of ϵ -optimal stopping times.

4.1 The Markov chain $(\Pi_n, S_n)_{n \geq 0}$

We start with some technical preliminary results that will be required in the sequel. We investigate the Markov property of the filter process and give details on the structure of the $(\mathfrak{F}_t^Y)_{t \geq 0}$ -stopping times.

Proposition 4.1. *The sequences $(\Pi_n, Y_n, S_n)_{n \in \mathbb{N}}$, $(\Pi_n, S_n)_{n \in \mathbb{N}}$ and $(\Pi_n)_{n \in \mathbb{N}}$ are $(\mathfrak{F}_{T_n}^Y)_{n \in \mathbb{N}}$ -Markov chains.*

Proof Let $h \in B(\mathcal{M}_1(E_0) \times \mathbb{R}^d \times \mathbb{R}^+)$. The law of iterated conditional expectations yields

$$\mathbf{E}[h(\Pi_n, Y_n, S_n) | \mathfrak{F}_{T_{n-1}}^Y] = \mathbf{E}[\mathbf{E}[h(\Pi_n, Y_n, S_n) | \mathfrak{F}_{T_{n-1}}] | \mathfrak{F}_{T_{n-1}}^Y].$$

From Proposition 3.4 and Eq. (1) which defines Y_n one obtains

$$\begin{aligned} & \mathbf{E}[h(\Pi_n, Y_n, S_n) | \mathfrak{F}_{T_{n-1}}] \\ &= \mathbf{E}\left[h\left(\Psi(\Pi_{n-1}, \varphi(Z_n) + W_n, S_n), \varphi(Z_n) + W_n, S_n\right) | \mathfrak{F}_{T_{n-1}}\right] \\ &= \sum_{j=1}^q \int h\left(\Psi(\Pi_{n-1}, \varphi(x_j) + w, s), \varphi(x_j) + w, s\right) \\ & \quad \times \mathbf{P}(Z_n = x_j, W_n \in dw, S_n \in ds | \mathfrak{F}_{T_{n-1}}). \end{aligned}$$

Yet, W_n is independent from $\sigma(Z_n, S_n) \vee \mathfrak{F}_{T_{n-1}}$ and admits the density function f_W . As in the proof of Lemma 3.1 one thus obtains

$$\begin{aligned} & \mathbf{E}[h(\Pi_n, Y_n, S_n) | \mathfrak{F}_{T_{n-1}}] \\ &= \sum_{j=1}^q \int h\left(\Psi(\Pi_{n-1}, y, s), y, s\right) \mathbf{P}(Z_n = x_j, S_n \in ds | \mathfrak{F}_{T_{n-1}}) f_W(y - \varphi(x_j)) dy. \end{aligned}$$

Besides, we have $\mathbf{P}(Z_n = x_j, S_n \in ds | \mathfrak{F}_{T_{n-1}}) = \mathbf{P}(Z_n = x_j, S_n \in ds | Z_{n-1})$ as in the proof of Lemma 3.3, so that one has

$$\begin{aligned} & \mathbf{E}[h(\Pi_n, Y_n, S_n) | \mathfrak{F}_{T_{n-1}}] \\ &= \sum_{i=1}^q \mathbb{1}_{\{Z_{n-1}=x_i\}} \sum_{j=1}^q \int \left(\int_0^{t_i^*} h\left(\Psi(\Pi_{n-1}, y, s), y, s\right) \lambda\left(\Phi(x_i, s)\right) e^{-\Lambda(x_i, s)} Q\left(\Phi(x_i, s), x_j\right) ds \\ & \quad + h\left(\Psi(\Pi_{n-1}, y, t_i^*), y, t_i^*\right) e^{-\Lambda(x_i, t_i^*)} Q\left(\Phi(x_i, t_i^*), x_j\right) \right) f_W(y - \varphi(x_j)) dy. \end{aligned}$$

Take now the conditional expectation w.r.t. $\mathfrak{F}_{T_{n-1}}^Y$, to obtain

$$\begin{aligned} & \mathbf{E}[h(\Pi_n, Y_n, S_n) | \mathfrak{F}_{T_{n-1}}^Y] \\ &= \sum_{i=1}^q \Pi_{n-1}^i \sum_{j=1}^q \int \left(\int_0^{t_i^*} h(\Psi(\Pi_{n-1}, y, s), y, s) \lambda(\Phi(x_i, s)) e^{-\Lambda(x_i, s)} Q(\Phi(x_i, s), x_j) ds \right. \\ & \quad \left. + h(\Psi(\Pi_{n-1}, y, t_i^*), y, t_i^*) e^{-\Lambda(x_i, t_i^*)} Q(\Phi(x_i, t_i^*), x_j) \right) f_W(y - \varphi(x_j)) dy. \end{aligned}$$

Hence $\mathbf{E}[h(\Pi_n, Y_n, S_n) | \mathfrak{F}_{T_{n-1}}^Y]$ is merely a function of Π_{n-1} yielding the result for the three processes. \square

4.2 The $(\mathfrak{F}_t^Y)_{t \geq 0}$ -stopping times

We now turn to the structure of the $(\mathfrak{F}_t^Y)_{t \geq 0}$ -stopping times.

Lemma 4.2. *For all $n \in \mathbb{N}$, T_n is an $(\mathfrak{F}_t^Y)_{t \geq 0}$ -stopping time.*

Proof Notice that for all $n \in \mathbb{N}$, $\mathbf{P}(Y_n = Y_{n+1}) = 0$. This stems from the absolute continuity of the distribution of the random variables $(W_n)_{n \in \mathbb{N}}$ since

$$\{Y_n = Y_{n+1}\} \subset \bigcup_{1 \leq i, j \leq q} \{W_n - W_{n+1} = \varphi(x_i) - \varphi(x_j)\}.$$

Hence, for all $n \in \mathbb{N}$ and $t \in \mathbb{R}^+$, one has \mathbf{P} a.s. $\{T_n \leq t\} = \{N_t \geq n\}$ where we denote $N_t = \sum_{0 \leq s \leq t} \mathbb{1}_{\{Y_s \neq Y_{s-}\}}$. The process $(N_t)_{t \geq 0}$ is \mathfrak{F}^Y -adapted thus $\{N_t \geq n\} \in \mathfrak{F}_t^Y$ and since the filtration \mathfrak{F}^Y contains the \mathbf{P} -null sets, one has $\{T_n \leq t\} \in \mathfrak{F}_t^Y$. For all $n \in \mathbb{N}$, T_n is therefore an $(\mathfrak{F}_t^Y)_{t \geq 0}$ -stopping time. \square

We now recall Theorem A2 T33 from [6] concerning the structure of the stopping times for point processes and apply it in our case.

Definition 4.3. *Define the filtration $(\mathfrak{F}_t^p)_{t \geq 0}$ as follows*

$$\mathfrak{F}_t^p = \sigma \left(\mathbb{1}_{\{Y_n \in A\}} \mathbb{1}_{\{T_n \leq s\}}; n \geq 1, 0 \leq s \leq t, A \in \mathcal{B}(\mathbb{R}^d) \right).$$

Theorem 4.4. *Let σ be an $(\mathfrak{F}_t^p)_{t \geq 0}$ -stopping time. For all $n \in \mathbb{N}$, there exists a $\mathfrak{F}_{T_n}^p$ -measurable non negative random variable R_n , such that one has*

$$\sigma \wedge T_{n+1} = (T_n + R_n) \wedge T_{n+1} \quad \text{on} \quad \{\sigma \geq T_n\}.$$

Our observation process $(Y_t)_{t \geq 0}$ being a point process that fits the framework developed in [6], we apply this Theorem to $(\mathfrak{F}_t^Y)_{t \geq 0}$ -stopping times.

Proposition 4.5. *For all $t \geq 0$, one has $\mathfrak{F}_t^Y = \mathfrak{F}_t^p$.*

Proof First prove that $\mathfrak{F}_t^Y \subset \mathfrak{F}_t^p$. Let $A \in \mathcal{B}(\mathbb{R}^d)$ and $0 \leq s \leq t$, one has

$$\{Y_s \in A\} = \bigcup_{n \in \mathbb{N}} \left(\{T_n \leq s < T_{n+1}\} \cap \{Y_n \in A\} \right) \in \mathfrak{F}_s^p \subset \mathfrak{F}_t^p.$$

Indeed, in the above equation, we used that T_0 and Y_0 are assumed to be deterministic. For the reverse inclusion, let $A \in \mathcal{B}(\mathbb{R}^d)$, $n \in \mathbb{N}^*$ and $0 \leq s \leq t$. Recall that $Y_n = Y_{T_n}$. One has $\{Y_{T_n} \in A\} \in \mathfrak{F}_{T_n}^Y$ since $(Y_t)_{t \geq 0}$ is \mathfrak{F}^Y -adapted and T_n is an $(\mathfrak{F}_t^Y)_{t \geq 0}$ -stopping time from Lemma 4.2. Therefore, one has $\{Y_n \in A\} \cap \{T_n \leq s\} \in \mathfrak{F}_s^Y \subset \mathfrak{F}_t^Y$, hence, the result. \square

We may therefore apply Theorem 4.4 to $(\mathfrak{F}_t^Y)_{t \geq 0}$ -stopping times.

Theorem 4.6. *Let σ be an $(\mathfrak{F}_t^Y)_{t \geq 0}$ -stopping time. For all $n \in \mathbb{N}$, there exists a non negative random variable R_n , $\mathfrak{F}_{T_n}^Y$ -measurable such that one has*

$$\sigma \wedge T_{n+1} = (T_n + R_n) \wedge T_{n+1} \quad \text{on} \quad \{\sigma \geq T_n\}.$$

We outline the following result, which is a direct consequence of the above theorem, because it will be used several times in our derivation.

Lemma 4.7. *For all $n \in \mathbb{N}$, $\{T_n \leq \sigma < T_{n+1}\} = \{T_n \leq \sigma\} \cap \{S_{n+1} > R_n\}$.*

Proof Theorem 4.6 states that on the event $\{T_n \leq \sigma\}$, one has $\sigma \wedge T_{n+1} = T_n + (R_n \wedge S_{n+1})$ so that, still on the event $\{T_n \leq \sigma\}$, one has $(\sigma < T_{n+1}) \Leftrightarrow (R_n < S_{n+1})$. We deduce the result from this observation. \square

We now investigate the effect of the translation operator of the Markov chain $(\Pi_n, Y_n, S_n)_{n \in \mathbb{N}}$ on the $(\mathfrak{F}_t^Y)_{t \geq 0}$ -stopping times. Proposition 4.1 states that $(\Pi_n, Y_n, S_n)_{n \in \mathbb{N}}$ is a $(\mathfrak{F}_{T_n}^Y)_{n \in \mathbb{N}}$ -Markov chain. Let us consider its canonical space $\Omega = (\mathcal{M}_1(E_0) \times \mathbb{R}^d \times \mathbb{R}^+)^{\mathbb{N}}$. Thus, for $\omega = (\omega_0, \omega_1, \dots) \in \Omega$, one has $(\Pi_n, Y_n, S_n)(\omega) = \omega_n$. Besides, we define the *translation operator*

$$\theta : \begin{cases} \Omega & \rightarrow \Omega \\ (\omega_0, \omega_1, \dots) & \rightarrow (\omega_1, \omega_2, \dots) \end{cases}$$

We then define $\theta^0 = Id_\Omega$ and recursively for $l \geq 2$, $\theta^l = \theta \circ \theta^{l-1}$. Thus, for all $n, l \in \mathbb{N}$, one has $(\Pi_n, Y_n, S_n) \circ \theta^l = (\Pi_{n+l}, Y_{n+l}, S_{n+l})$. As $T_0 = 0$, one has

$$T_n \circ \theta^l = \sum_{k=1}^n S_k \circ \theta^l = \sum_{k=1}^n S_{k+l} = T_{n+l} - T_l.$$

The next results of this section, Lemma 4.8, Propositions 4.11 and 4.12 and Corollary 4.13, are given without proof because their proofs follow the very same lines as in [5] from which they are adapted. However, it is important to notice that the results from [5] cannot be applied directly to our case because the sequence $(\Pi_n, Y_n, S_n)_{n \in \mathbb{N}}$, although it is a Markov chain, is not the underlying Markov chain of some PDMP. Set now $\sigma \in \Sigma^Y$. From Theorem 4.6, for all $n \in \mathbb{N}$, there exists a non negative $\mathfrak{F}_{T_n}^Y$ -measurable random variable R_n , such that, on the event $\{\sigma \geq T_n\}$, one has $\sigma \wedge T_{n+1} = (T_n + R_n) \wedge T_{n+1}$.

Lemma 4.8. Let $\bar{R}_0 = R_0$ and for $k \geq 1$, $\bar{R}_k = R_k \mathbb{1}_{\{S_k \leq \bar{R}_{k-1}\}}$. One has then

$$\sigma = \sum_{n=1}^{\infty} \bar{R}_{n-1} \wedge S_n.$$

Remark 4.9. This lemma proves that in Theorem 4.6, the sequence $(R_n)_{n \in \mathbb{N}}$ can be replaced by $(\bar{R}_n)_{n \in \mathbb{N}}$. Therefore, in the sequel, we will assume, without loss of generality that the sequence $(R_n)_{n \in \mathbb{N}}$ satisfies the following condition: for all $n \in \mathbb{N}$, $R_{n+1} = 0$ on the event $\{S_{n+1} > R_n\}$.

There exists a sequence of real-valued measurable functions $(r_k)_{k \in \mathbb{N}}$ defined on $(\mathbb{R}^d \times \mathbb{R}^+)^{k+1}$ such that $R_k = r_k(\mathcal{G}_k)$, where $\mathcal{G}_k = (Y_0, S_0, \dots, Y_k, S_k)$. Indeed, $\mathfrak{F}_{T_k}^Y = \sigma(Y_j, S_j, j \leq k)$ and R_k is $\mathfrak{F}_{T_k}^Y$ -measurable.

Definition 4.10. Let $l \in \mathbb{N}^*$ and $(\tilde{R}_k^l)_{k \in \mathbb{N}}$ be a sequence of functions defined on $(\mathbb{R}^d \times \mathbb{R}^+)^{l+1} \times \Omega$ by $\tilde{R}_0^l(\gamma, \omega) = r_l(\gamma)$ and for $k \geq 1$, $\tilde{R}_k^l(\gamma, \omega) = r_{l+k}(\gamma, \mathcal{G}_{k-1}(\omega)) \mathbb{1}_{\{S_k \leq \tilde{R}_{k-1}^l\}}(\gamma, \omega)$.

Proposition 4.11. Assume that $T_l \leq \sigma \leq T_N$. For all $k \in \mathbb{N}$, one has then $\tilde{R}_k^l(\mathcal{G}_l, \theta^l) = \bar{R}_{l+k}$ and $\sigma = T_l + \tilde{\sigma}(\mathcal{G}_l, \theta^l)$, with $\tilde{\sigma} : (\mathbb{R}^d \times \mathbb{R}^+)^{l+1} \times \Omega \rightarrow \mathbb{R}^+$ defined as $\tilde{\sigma}(\gamma, \omega) = \sum_{n=1}^{N-l} \tilde{R}_{n-1}^l(\gamma, \omega) \wedge S_n(\omega)$.

Proposition 4.12. Let $(U_n)_{n \in \mathbb{N}}$ be a sequence of non negative random variables such that for all n , U_n is $\mathfrak{F}_{T_n}^Y$ -measurable and $U_{n+1} = 0$ on $\{S_{n+1} > U_n\}$. We define $U = \sum_{n=1}^{\infty} U_{n-1} \wedge S_n$. Then U is an $(\mathfrak{F}_t^Y)_{t \geq 0}$ -stopping time.

Corollary 4.13. For all $\gamma \in (\mathbb{R}^d \times \mathbb{R}^+)^{p+1}$, $\tilde{\sigma}(\gamma, \cdot)$ is a $(\mathfrak{F}_t^Y)_{t \geq 0}$ -stopping time.

4.3 Optimal stopping problem under complete observation

In this section, we show how our optimal stopping problem under partial observation for the process $(X_t)_{t \geq 0}$ can be converted into an optimal stopping problem under complete observation involving the discrete-time Markov chain $(\Pi_n, S_n)_{0 \leq n \leq N}$. However, this does not correspond to the optimal stopping problem under complete observation for PDMP's studied in [4, 5] because the Markov chain $(\Pi_n, S_n)_{n \geq 0}$ is not the underlying Markov chain of some PDMP.

Lemma 4.14. Let $\sigma \in \Sigma^Y$ and $n \geq 1$. For all $\pi \in \mathcal{M}_1(E_0)$ one has

$$\begin{aligned} & \mathbf{E}[g(X_{\sigma \wedge T_n}) | \Pi_0 = \pi] \\ &= \sum_{k=0}^{n-1} \sum_{i=1}^q \mathbf{E}[\mathbb{1}_{\{T_k \leq \sigma\}} \mathbb{1}_{\{R_k < t_i^*\}} g \circ \Phi(x_i, R_k) e^{-\Lambda(x_i, R_k)} \Pi_k^i | \Pi_0 = \pi] \\ & \quad + \sum_{i=1}^q \mathbf{E}[\mathbb{1}_{\{T_n \leq \sigma\}} g(x_i) \Pi_n^i | \Pi_0 = \pi]. \end{aligned}$$

Proof We split $\mathbf{E}[g(X_{\sigma \wedge T_n}) | \Pi_0 = \pi]$ into several terms depending on the position of σ w.r.t. the jump times T_k

$$\begin{aligned} \mathbf{E}[g(X_{\sigma \wedge T_n}) | \Pi_0 = \pi] &= \sum_{k=0}^{n-1} \sum_{i=1}^q \mathbf{E}[\mathbb{1}_{\{T_k \leq \sigma < T_{k+1}\}} \mathbb{1}_{\{Z_k = x_i\}} g \circ \Phi(x_i, R_k) | \Pi_0 = \pi] \\ &\quad + \sum_{i=1}^q \mathbf{E}[\mathbb{1}_{\{T_n \leq \sigma\}} \mathbb{1}_{\{Z_n = x_i\}} g(x_i) | \Pi_0 = \pi]. \end{aligned}$$

For notational convenience, consider

$$\begin{cases} A_{k,i} &= \mathbb{1}_{\{T_k \leq \sigma < T_{k+1}\}} \mathbb{1}_{\{Z_k = x_i\}} g \circ \Phi(x_i, R_k), \\ B_i &= \mathbb{1}_{\{T_n \leq \sigma\}} \mathbb{1}_{\{Z_n = x_i\}} g(x_i). \end{cases}$$

On the one hand, one has $\mathbf{E}[B_i | \mathfrak{F}_{T_n}^Y] = g(x_i) \mathbb{1}_{\{T_n \leq \sigma\}} \Pi_n^i$ since $\{T_n \leq \sigma\} \in \mathfrak{F}_{T_n}^Y$ (see for instance [6, p. 298, Theorem T7]). On the other hand, to compute $\mathbf{E}[A_{k,i} | \mathfrak{F}_{T_k}^Y]$, we use Lemma 4.7 to obtain

$$\begin{aligned} \mathbf{E}[A_{k,i} | \mathfrak{F}_{T_k}^Y] &= \mathbb{1}_{\{T_k \leq \sigma\}} g \circ \Phi(x_i, R_k) \mathbf{E}[\mathbb{1}_{\{S_{k+1} > R_k\}} \mathbb{1}_{\{Z_k = x_i\}} | \mathfrak{F}_{T_k}^Y] \\ &= \mathbb{1}_{\{T_k \leq \sigma\}} g \circ \Phi(x_i, R_k) \mathbf{E}[\mathbb{1}_{\{Z_k = x_i\}} \mathbf{E}[\mathbb{1}_{\{S_{k+1} > R_k\}} | \mathfrak{F}_{T_k}] | \mathfrak{F}_{T_k}^Y] \\ &= \mathbb{1}_{\{T_k \leq \sigma\}} g \circ \Phi(x_i, R_k) \mathbf{E}[\mathbb{1}_{\{Z_k = x_i\}} \mathbb{1}_{\{R_k < t^*(Z_k)\}} e^{-\Lambda(Z_k, R_k)} | \mathfrak{F}_{T_k}^Y] \\ &= \mathbb{1}_{\{T_k \leq \sigma\}} g \circ \Phi(x_i, R_k) \mathbb{1}_{\{R_k < t_i^*\}} e^{-\Lambda(x_i, R_k)} \Pi_k^i. \end{aligned}$$

Details to obtain the third line in the above computations are provided by Lemma A.2. The result follows. \square

4.4 Dynamic programming operators

Based on the decomposition given by Lemma 4.14, we introduce the dynamic programming operators for the optimal stopping problem (3).

Definition 4.15. *The operator $H : B(E) \rightarrow B(\mathcal{M}_1(E_0) \times \mathbb{R}^+)$ is defined for all $h \in B(E)$ and $(\pi, u) \in \mathcal{M}_1(E_0) \times \mathbb{R}^+$ by*

$$Hh(\pi, u) = \mathbf{E}[h \circ \Phi(Z_0, u) \mathbb{1}_{\{S_1 > u\}} | \Pi_0 = \pi].$$

The following lemma is straightforward from the distribution of S_1 .

Lemma 4.16. *For all $h \in B(E)$ and $(\pi, u) \in \mathcal{M}_1(E_0) \times \mathbb{R}^+$, one has*

$$Hh(\pi, u) = \sum_{i=1}^q \pi^i \mathbb{1}_{\{u < t_i^*\}} e^{-\Lambda(x_i, u)} h \circ \Phi(x_i, u).$$

The function $u \rightarrow Hh(\pi, u)$ is right continuous with left limits, is continuous on the intervals $]t_m^*; t_{m+1}^*[$ for all $m \in \{0, \dots, q-1\}$ and is null for $u \geq t_q^*$. For all $h \in B(E)$, we consider the restriction $H^m h$ of Hh to $\mathcal{M}_1(E_0) \times [t_m^*; t_{m+1}^*[$ extended continuously by constants to $\mathcal{M}_1(E_0) \times \mathbb{R}^+$.

Definition 4.17. For all $m \in \{0, \dots, q-1\}$, we define the operator $H^m: B(E) \rightarrow B(\mathcal{M}_1(E_0) \times \mathbb{R}^+)$ as follows

- if $u < t_m^*$, $H^m h(\pi, u) = Hh(\pi, t_m^*)$,
- if $u \geq t_m^*$, $H^m h(\pi, u) = \sum_{i=m+1}^q \pi^i e^{-\Lambda(x_i, u \wedge t_{m+1}^*)} h \circ \Phi(x_i, u \wedge t_{m+1}^*)$.

Remark 4.18. For all $m \in \{0, \dots, q-1\}$ and for all $h \in B(E)$, $\pi \in \mathcal{M}_1(E_0)$, the function $u \rightarrow H^m h(\pi, u)$ is continuous. Moreover, it is constant on $[0; t_m^*]$ and on $[t_{m+1}^*; +\infty[$ and one has $Hh(\pi, u) = \sum_{m=0}^{q-1} \mathbb{1}_{[t_m^*, t_{m+1}^*]}(u) H^m h(\pi, u)$.

Definition 4.19. The operators $I: B(\mathcal{M}_1(E_0)) \rightarrow B(\mathcal{M}_1(E_0) \times \mathbb{R}^+)$, $G: B(\mathcal{M}_1(E_0)) \rightarrow B(\mathcal{M}_1(E_0) \times \mathbb{R}^+)$ and $K: B(\mathcal{M}_1(E_0)) \rightarrow B(\mathcal{M}_1(E_0))$ are defined for all $v \in B(\mathcal{M}_1(E_0))$ and $(\pi, u) \in \mathcal{M}_1(E_0) \times \mathbb{R}^+$ by

$$\begin{aligned} Iv(\pi, u) &= \mathbf{E}[v(\Pi_1) \mathbb{1}_{\{S_1 < u \wedge t^*(Z_0)\}} | \Pi_0 = \pi], \\ Gv(\pi, u) &= \mathbf{E}[v(\Pi_1) \mathbb{1}_{\{S_1 \leq u\}} | \Pi_0 = \pi], \\ Kv(\pi) &= \mathbf{E}[v(\Pi_1) | \Pi_0 = \pi] = Gv(\pi, t_q^*). \end{aligned}$$

Computations similar to the ones led in the proof of Proposition 4.1 yield developed forms for these operators.

Lemma 4.20. For all $v \in B(\mathcal{M}_1(E_0))$ and $(\pi, u) \in \mathcal{M}_1(E_0) \times \mathbb{R}^+$, one has

$$\begin{aligned} Iv(\pi, u) &= \sum_{i=1}^q \pi^i \int_0^{u \wedge t_i^*} \left(\lambda \circ \Phi(x_i, s') e^{-\Lambda(x_i, s')} \right. \\ &\quad \times \left. \int_{\mathbb{R}^d} v(\Psi(\pi, y', s')) \sum_{j=1}^q Q(\Phi(x_i, s'), x_j) f_W(y' - \varphi(x_j)) dy' \right) ds', \\ Gv(\pi, u) &= Iv(\pi, u) + \sum_{i=1}^q \pi^i \mathbb{1}_{\{t_i^* \leq u\}} e^{-\Lambda(x_i, t_i^*)} \\ &\quad \times \int_{\mathbb{R}^d} v(\Psi(\pi, y', t_i^*)) \sum_{j=1}^q Q(\Phi(x_i, t_i^*), x_j) f_W(y' - \varphi(x_j)) dy'. \end{aligned}$$

As for operator H , we split G into a sum of continuous operators.

Definition 4.21. For all $m \in \{0, \dots, q-1\}$, we define the operator $G^m: B(\mathcal{M}_1(E_0)) \rightarrow B(\mathcal{M}_1(E_0) \times \mathbb{R}^+)$ as $G^m v(\pi, u) = Gv(\pi, t_m^*)$ if $u < t_m^*$ and

$$\begin{aligned} G^m v(\pi, u) &= Iv(\pi, u \wedge t_{m+1}^*) + \sum_{i=1}^m \pi^i e^{-\Lambda(x_i, t_i^*)} \\ &\quad \times \int_{\mathbb{R}^d} v(\Psi(\pi, y', t_i^*)) \sum_{j=1}^q Q(\Phi(x_i, t_i^*), x_j) f_W(y' - \varphi(x_j)) dy' \end{aligned}$$

if $u \geq t_m^*$. In particular, one has

$$Gv(\pi, u) = \sum_{m=0}^{q-1} \mathbb{1}_{[t_m^*, t_{m+1}^*]}(u) G^m v(\pi, u) + \mathbb{1}_{[t_q^*, +\infty]}(u) Kv(\pi).$$

Definition 4.22. The operators $J: B(\mathcal{M}_1(E_0)) \times B(E) \rightarrow B(\mathcal{M}_1(E_0) \times \mathbb{R}^+)$, $J^m: B(\mathcal{M}_1(E_0)) \times B(E) \rightarrow B(\mathcal{M}_1(E_0) \times \mathbb{R}^+)$ for all $m \in \{0, \dots, q-1\}$ and $L: B(\mathcal{M}_1(E_0)) \times B(E) \rightarrow B(\mathcal{M}_1(E_0))$ are defined for all $(v, h) \in B(\mathcal{M}_1(E_0)) \times B(E)$ and $(\pi, u) \in \mathcal{M}_1(E_0) \times \mathbb{R}^+$ by

$$\begin{aligned} J^m(v, h)(\pi, u) &= H^m h(\pi, u) + G^m v(\pi, u), \\ J(v, h)(\pi, u) &= H h(\pi, u) + G v(\pi, u), \\ L(v, h)(\pi) &= \sup_{u \geq 0} J(v, h)(\pi, u). \end{aligned}$$

In particular, one has

$$J(v, h)(\pi, u) = \sum_{m=0}^{q-1} \mathbb{1}_{[t_m^*, t_{m+1}^*]}(u) J^m(v, h)(\pi, u) + \mathbb{1}_{[t_q^*, +\infty)}(u) K v(\pi).$$

In view of the above definitions, it seems natural to distinguish whether $t_m^* = t_{m+1}^*$ or $t_m^* < t_{m+1}^*$.

Definition 4.23. Let $M \subset \{0, \dots, q-1\}$ be the set of indices m such that $t_m^* < t_{m+1}^*$.

Notice that M is not empty because it contains at least the index 0 since we assumed that $t_1^* > 0 = t_0^*$. As a straightforward consequence of the previous definitions, one has the following result.

Lemma 4.24. For $(v, h) \in B(\mathcal{M}_1(E_0)) \times B(E)$ and $\pi \in \mathcal{M}_1(E_0)$, one has

$$L(v, h)(\pi) = \max_{m \in M} \left\{ \sup_{u \in [t_m^*, t_{m+1}^*]} J^m(v, h)(\pi, u) \right\} \vee K v(\pi).$$

Definition 4.25. We define recursively the sequence $(v_n)_{0 \leq n \leq N}$ of functions from $\mathcal{M}_1(E_0)$ onto \mathbb{R} as follows

$$\begin{cases} v_N(\pi) &= \sum_{i=1}^q g(x_i) \pi^i, \\ v_{n-1}(\pi) &= L(v_n, g)(\pi), \quad 1 \leq n \leq N. \end{cases}$$

The rest of this section is dedicated to proving that v_n is the value function of the optimal stopping problem with horizon T_{N-n} .

4.5 Recursive construction of the value function and ϵ -optimal stopping times

The two following theorems 4.26 and 4.29 establish that v_n defined above is the value function of our partially observed optimal stopping problem with horizon T_{N-n} . Moreover, Theorem 4.29 gives an explicit construction of a family of ϵ -optimal stopping times. This section is adapted from [4].

Theorem 4.26. For all $1 \leq n \leq N$ and $\pi \in \mathcal{M}_1(E_0)$, one has

$$\sup_{\sigma \in \Sigma_n^X} \mathbf{E}[g(X_\sigma) | \Pi_0 = \pi] \leq v_{N-n}(\pi).$$

Proof Let $\sigma \in \Sigma^Y$. We prove the theorem by induction. For $n = 1$, Lemma 4.14 yields

$$\begin{aligned} \mathbf{E}[g(X_{\sigma \wedge T_1}) | \Pi_0 = \pi] &= \sum_{i=1}^q \mathbf{E}[\mathbb{1}_{\{R_0 < t_i^*\}} g \circ \Phi(x_i, R_0) e^{-\Lambda(x_i, R_0)} \Pi_0^i | \Pi_0 = \pi] \\ &\quad + \sum_{i=1}^q \mathbf{E}[\mathbb{1}_{\{T_1 \leq \sigma\}} g(x_i) \Pi_1^i | \Pi_0 = \pi] = (a) + (b). \end{aligned}$$

Since R_0 is deterministic, we recognize, in the term (a), the form given in Lemma 4.16) of operator H , thus $(a) = Hg(\pi, R_0)$. We now turn to the term (b). From Lemma 4.7 one has

$$\begin{aligned} (b) &= \mathbf{E}[\mathbb{1}_{\{S_1 \leq R_0\}} \sum_{i=1}^q g(x_i) \Pi_1^i | \Pi_0 = \pi] \\ &= \mathbf{E}[v_N(\Pi_1) \mathbb{1}_{\{S_1 \leq R_0\}} | \Pi_0 = \pi] = Gv_N(\pi, R_0). \end{aligned}$$

Recall that from Definition 4.22 one has $J(v_N, g) = Hg + Gv_N$ thus, adding (a) and (b), one obtains

$$\begin{aligned} \mathbf{E}[g(X_{\sigma \wedge T_1}) | \Pi_0 = \pi] &= J(v_N, g)(\pi, R_0) \leq \sup_{u \geq 0} J(v_N, g)(\pi, u) \\ &= L(v_N, g)(\pi) = v_{N-1}(\pi). \end{aligned}$$

Set now $2 \leq n \leq N$ and assume that for all $\tau \in \Sigma_{n-1}^Y$, one has $\mathbf{E}[g(X_\tau) | \Pi_0 = \pi] \leq v_{N-(n-1)}(\pi)$. Lemma 4.14 yields

$$\begin{aligned} &\mathbf{E}[g(X_{\sigma \wedge T_n}) | \Pi_0 = \pi] \\ &= \sum_{k=0}^{n-1} \sum_{i=1}^q \mathbf{E}[\mathbb{1}_{\{T_k \leq \sigma\}} \mathbb{1}_{\{R_k < t_i^*\}} g \circ \Phi(x_i, R_k) e^{-\Lambda(x_i, R_k)} \Pi_k^i | \Pi_0 = \pi] \\ &\quad + \sum_{i=1}^q \mathbf{E}[\mathbb{1}_{\{T_n \leq \sigma\}} g(x_i) \Pi_n^i | \Pi_0 = \pi]. \end{aligned}$$

As in the case $n = 1$, the term for $k = 0$ equals $Hg(\pi, R_0)$. Notice that the other terms are null on $\{T_1 > \sigma\}$, thus we may factorize $\mathbb{1}_{\{T_1 \leq \sigma\}} = \mathbb{1}_{\{S_1 \leq R_0\}}$ that is $\mathfrak{F}_{T_1}^Y$ -measurable. Take now the conditional expectation w.r.t. $\mathfrak{F}_{T_1}^Y$ in these terms to obtain

$$\mathbf{E}[g(X_{\sigma \wedge T_n}) | \Pi_0 = \pi] = Hg(\pi, R_0) + \mathbf{E}[\Xi \mathbb{1}_{\{S_1 \leq R_0\}} | \Pi_0 = \pi], \quad (8)$$

where we defined Ξ as

$$\begin{aligned} \Xi &= \mathbf{E} \left[\sum_{k=1}^{n-1} \sum_{i=1}^q \mathbb{1}_{\{T_k \leq \sigma\}} \mathbb{1}_{\{R_k < t_i^*\}} g \circ \Phi(x_i, R_k) e^{-\Lambda(x_i, R_k)} \Pi_k^i \right. \\ &\quad \left. + \sum_{i=1}^q \mathbb{1}_{\{T_n \leq \sigma\}} g(x_i) \Pi_n^i \middle| \mathfrak{F}_{T_1}^Y \right]. \end{aligned}$$

We now use the Markov property of the chain $(\Pi_k)_{k \geq 0}$. Indeed, for $k \geq 1$, one has $\Pi_k = \Pi_{k-1} \circ \theta$. Moreover, when $T_1 \leq \sigma$, one has, from Proposition 4.11, $R_k = \tilde{R}_{k-1}^1 \circ \theta$ (indeed, we pointed out in Remark 4.9 that R_k can be replaced by \bar{R}_k defined in

Lemma 4.8) and $\sigma = T_1 + \tilde{\sigma} \circ \theta$ where \tilde{R}_{k-1}^1 and $\tilde{\sigma}$ are defined in Definition 4.10 and Proposition 4.11 (with $l = 1$ in the present case). Since for $k \geq 1$, $T_k = T_1 + T_{k-1} \circ \theta$, one has $\mathbb{1}_{\{T_k \leq \sigma\}} = \mathbb{1}_{\{T_{k-1} \leq \tilde{\sigma}\}} \circ \theta$. Finally, the Markov property of the chain $(\Pi_k)_{k \geq 0}$ yields

$$\begin{aligned} \Xi &= \mathbf{E}_{\Pi_1} \left[\sum_{k=0}^{n-2} \sum_{i=1}^q \mathbb{1}_{\{T_k \leq \tilde{\sigma}\}} \mathbb{1}_{\{\tilde{R}_k^1 < t_i^*\}} g \circ \Phi(x_i, \tilde{R}_k^1) e^{-\Lambda(x_i, \tilde{R}_k^1)} \Pi_k^i \right. \\ &\quad \left. + \sum_{i=1}^q \mathbb{1}_{\{T_{n-1} \leq \tilde{\sigma}\}} g(x_i) \Pi_{n-1}^i \right]. \end{aligned}$$

In other words, define $w(\pi) = \mathbf{E}[g(X_{\tilde{\sigma} \wedge T_{n-1}}) | \Pi_0 = \pi]$. Using Lemma 4.14, we recognize that $\Xi = w(\Pi_1)$. Moreover, one has $w(\pi) \leq v_{N-(n-1)}(\pi)$ from the induction assumption since $\tilde{\sigma} \wedge T_{n-1} \in \Sigma_{n-1}^Y$ (indeed, both $\tilde{\sigma}$ and T_{n-1} are $(\mathfrak{F}_t^Y)_{t \geq 0}$ -stopping times from Corollary 4.13 and Proposition 4.2 respectively). One has then

$$\Xi \leq v_{N-(n-1)}(\Pi_1). \quad (9)$$

Finally, combining Eq. (8) and (9), one has

$$\mathbf{E}[g(X_{\sigma \wedge T_n}) | \Pi_0 = \pi] \leq Hg(\pi, R_0) + \mathbf{E}[v_{N-(n-1)}(\Pi_1) \mathbb{1}_{\{S_1 \leq R_0\}} | \Pi_0 = \pi].$$

In the second term, we recognize the operator G and one has

$$\begin{aligned} \mathbf{E}[g(X_{\sigma \wedge T_n}) | \Pi_0 = \pi] &\leq Hg(\pi, R_0) + Gv_{N-(n-1)}(\pi, R_0) \\ &= J(v_{N-(n-1)}, g)(\pi, R_0) \\ &\leq \sup_{u \geq 0} J(v_{N-(n-1)}, g)(\pi, u) \\ &= L(v_{N-(n-1)}, g)(\pi) = v_{N-n}(\pi), \end{aligned}$$

that proves the induction. \square

Theorem 4.26 proves that v_n is an upper bound for the value function of the problem with horizon T_{N-n} . We now prove the reverse inequality by constructing a sequence of ϵ -optimal stopping times.

Definition 4.27. For $\epsilon > 0$, $1 \leq n \leq N$ and for $\pi \in \mathcal{M}_1(E_0)$, we define

$$r_n^\epsilon(\pi) = \inf \{u > 0 : J(v_{N-n}, g)(\pi, u) > v_{N-n-1}(\pi) - \epsilon\}.$$

Consider $R_{1,0}^\epsilon = r_0^\epsilon(\Pi_0)$ and for $2 \leq n \leq N$,

$$\begin{cases} R_{n,0}^\epsilon &= r_{n-1}^{\epsilon/2}(\Pi_0), \\ R_{n,k}^\epsilon &= r_{n-1-k}^{\epsilon/(2^{k+1})}(\Pi_k) \mathbb{1}_{\{R_{n,k-1}^\epsilon \geq S_k\}} \text{ for } 1 \leq k \leq n-2, \\ R_{n,n-1}^\epsilon &= r_0^{\epsilon/(2^{n-1})}(\Pi_{n-1}) \mathbb{1}_{\{R_{n,n-2}^\epsilon \geq S_{n-1}\}}, \end{cases}$$

and finally set

$$U_n^\epsilon = \sum_{k=1}^n R_{n,k-1}^\epsilon \wedge S_k.$$

The following lemma concerns the effect of the translation operator θ on the sequence $(R_{n,k}^\epsilon)_{1 \leq n \leq N, 0 \leq k \leq n-1}$.

Lemma 4.28. For $n \geq 2$ and $1 \leq k \leq n-1$, on the set $\{T_1 \leq U_n^{2\epsilon}\}$, one has

$$R_{n-1,k-1}^\epsilon \circ \theta = R_{n,k}^{2\epsilon}.$$

Proof For $n = 2$, one just has to prove that on the event $\{T_1 \leq U_2^{2\epsilon}\}$, one has $R_{1,0}^\epsilon \circ \theta = R_{2,1}^{2\epsilon}$. Yet, from the definition of the sequence $(R_{n,k}^\epsilon)_{1 \leq n \leq N, 0 \leq k \leq n-1}$, one has $R_{1,0}^\epsilon \circ \theta = r_0^\epsilon(\Pi_1)$ and $R_{2,1}^{2\epsilon} = r_0^{\frac{2\epsilon}{2}}(\Pi_1) \mathbb{1}_{\{R_{2,0}^{2\epsilon} \geq S_1\}}$. The result follows since we are on the event $\{T_1 \leq U_2^{2\epsilon}\} = \{R_{2,0}^{2\epsilon} \geq S_1\}$. For a fixed $n \geq 3$, we prove the lemma by induction on $1 \leq k \leq n-1$. Set $k = 1$. One has from the definition on the sequence $(R_{n,k}^\epsilon)_{1 \leq n \leq N, 0 \leq k \leq n-1}$, $R_{n-1,0}^\epsilon \circ \theta = r_{n-2}^{\frac{\epsilon}{2}}(\Pi_1)$ and $R_{n,1}^{2\epsilon} = r_{n-2}^{\frac{2\epsilon}{4}}(\Pi_1) \mathbb{1}_{\{R_{n,0}^{2\epsilon} \geq S_1\}}$. We obtain $R_{n-1,0}^\epsilon \circ \theta = R_{n,1}^{2\epsilon}$ because we have assumed that we are on the event $\{T_1 \leq U_n^{2\epsilon}\} = \{R_{n,0}^{2\epsilon} \geq S_1\}$. The propagation of the induction is similar to the case $k = 1$. \square

Equipped with this preliminary result, we may now prove that $(U_n^\epsilon)_{1 \leq n \leq N}$ is a sequence of ϵ -optimal stopping times for the observations filtration.

Theorem 4.29. *For all $1 \leq n \leq N$ and $\epsilon > 0$, one has $U_n^\epsilon \in \Sigma_n^Y$ and*

$$\mathbf{E}[g(X_{U_n^\epsilon}) | \Pi_0 = \pi] \geq v_{N-n}(\pi) - \epsilon.$$

Proof Let $n \in \{1, \dots, N\}$. First notice that, as a direct consequence of Proposition 4.12, U_n^ϵ is an $(\mathfrak{F}_t^Y)_{t \geq 0}$ -stopping time since, by construction, the $R_{n,k}^\epsilon$ are $\mathfrak{F}_{T_k}^Y$ -measurable and satisfy the condition $R_{n,k}^\epsilon = 0$ on the event $\{S_k > R_{n,k-1}^\epsilon\}$. It is also clear that $U_n^\epsilon \leq \sum_{k=1}^n S_k = T_n$. Thus, one has $U_n^\epsilon \in \Sigma_n^Y$. Let us now prove the second assessment by induction. Set $n = 1$. Let $\pi \in \mathcal{M}_1(E_0)$, we denote $r_0^\epsilon = r_0^\epsilon(\pi)$. Since $R_{1,0}^\epsilon = r_0^\epsilon$ is deterministic, one has clearly $R_{1,0}^\epsilon \in \Sigma^Y$ and we may apply Lemma 4.14 to $\sigma = R_{1,0}^\epsilon$ (and $n = 1$) which yields

$$\begin{aligned} \mathbf{E}[g(X_{R_{1,0}^\epsilon \wedge S_1}) | \Pi_0 = \pi] &= \sum_{i=1}^q \mathbf{E}[\mathbb{1}_{\{r_0^\epsilon < t_i^*\}} g \circ \Phi(x_i, r_0^\epsilon) e^{-\Lambda(x_i, r_0^\epsilon)} \Pi_0^i | \Pi_0 = \pi] \\ &\quad + \sum_{i=1}^q \mathbf{E}[\mathbb{1}_{\{S_1 \leq r_0^\epsilon\}} g(x_i) \Pi_1^i | \Pi_0 = \pi] = (c) + (d). \end{aligned}$$

As in the proof of the previous theorem, we recognize respectively the operators H and G in the terms (c) and (d). More precisely, $(c) = Hg(\pi, r_0^\epsilon)$ and

$$(d) = \mathbf{E}[\mathbb{1}_{\{S_1 \leq r_0^\epsilon\}} \sum_{i=1}^q g(x_i) \Pi_1^i | \Pi_0 = \pi] = \mathbf{E}[\mathbb{1}_{\{S_1 \leq r_0^\epsilon\}} v_N(\Pi_1) | \Pi_0 = \pi] = Gv_N(\pi, r_0^\epsilon),$$

so that, adding (c) and (d) yields $\mathbf{E}[g(X_{R_{1,0}^\epsilon \wedge S_1}) | \Pi_0 = \pi] = J(v_N, g)(\pi, r_0^\epsilon)$. Finally, the definition of r_0^ϵ yields $J(v_N, g)(\pi, r_0^\epsilon) \geq v_{N-1}(\pi) - \epsilon$ thus one has

$$\mathbf{E}[g(X_{R_{1,0}^\epsilon \wedge S_1}) | \Pi_0 = \pi] \geq v_{N-1}(\pi) - \epsilon.$$

Now set $2 \leq n \leq N$ and assume that for all $\epsilon > 0$, one has $\mathbf{E}[g(X_{U_{n-1}^\epsilon}) | \Pi_0 = \pi] \geq v_{N-(n-1)}(\pi) - \epsilon$. Lemma 4.14 yields

$$\begin{aligned} &\mathbf{E}[g(X_{U_n^{2\epsilon}}) | \Pi_0 = \pi] \\ &= \sum_{k=0}^{n-1} \sum_{i=1}^q \mathbf{E}[\mathbb{1}_{\{T_k \leq U_n^{2\epsilon}\}} \mathbb{1}_{\{R_{n,k}^{2\epsilon} < t_i^*\}} g \circ \Phi(x_i, R_{n,k}^{2\epsilon}) e^{-\Lambda(x_i, R_{n,k}^{2\epsilon})} \Pi_k^i | \Pi_0 = \pi] \\ &\quad + \sum_{i=1}^q \mathbf{E}[\mathbb{1}_{\{T_n \leq U_n^{2\epsilon}\}} g(x_i) \Pi_n^i | \Pi_0 = \pi]. \end{aligned}$$

Denote $r_{n-1}^\epsilon = r_{n-1}^\epsilon(\pi)$. As in the case $n = 1$, the term for $k = 0$ equals $Hg(\pi, r_{n-1}^\epsilon)$ since $R_{n,0}^{2\epsilon} = r_{n-1}^\epsilon(\Pi_0)$. Take the conditional expectation w.r.t. $\mathfrak{F}_{T_1}^Y$ in the other terms. One has then,

$$\mathbf{E}[g(X_{U_n^{2\epsilon}})|\Pi_0 = \pi] = Hg(\pi, r_{n-1}^\epsilon) + \mathbf{E}[\Xi' \mathbb{1}_{\{T_1 \leq U_n^{2\epsilon}\}}|\Pi_0 = \pi], \quad (10)$$

with

$$\begin{aligned} \Xi' = & \mathbf{E} \left[\sum_{k=1}^{n-1} \sum_{i=1}^q \mathbb{1}_{\{T_k \leq U_n^{2\epsilon}\}} \mathbb{1}_{\{R_{n,k}^{2\epsilon} < t_i^*\}} g \circ \Phi(x_i, R_{n,k}^{2\epsilon}) e^{-\Lambda(x_i, R_{n,k}^{2\epsilon})} \Pi_k^i \right. \\ & \left. + \sum_{i=1}^q \mathbb{1}_{\{T_n \leq U_n^{2\epsilon}\}} g(x_i) \Pi_n^i \middle| \mathfrak{F}_{T_1}^Y \right]. \end{aligned}$$

We wish to apply the Markov property of $(\Pi_k)_{k \in \mathbb{N}}$ in the term Ξ' . Recall that, from Lemma 4.28, one has $R_{n-1,k-1}^\epsilon \circ \theta = R_{n,k}^{2\epsilon}$ for $n \geq 2$ and $1 \leq k \leq n-1$ on the event $\{T_1 \leq U_n^{2\epsilon}\} = \{S_1 \leq R_{n,0}^{2\epsilon}\}$ (the equality of these events stems from Lemma 4.7). Thus, on this set one has

$$\begin{aligned} U_n^{2\epsilon} &= S_1 + \sum_{k=2}^n R_{n,k-1}^{2\epsilon} \wedge S_k = T_1 + \sum_{k=2}^n (R_{n-1,k-2}^\epsilon \circ \theta) \wedge (S_{k-1} \circ \theta) \\ &= T_1 + U_{n-1}^\epsilon \circ \theta. \end{aligned}$$

Besides, recall that for $k \geq 1$, $T_k = T_1 + T_{k-1} \circ \theta$ and thus one has $\mathbb{1}_{\{T_k \leq U_n^{2\epsilon}\}} = \mathbb{1}_{\{T_{k-1} \leq U_{n-1}^\epsilon\}} \circ \theta$. Therefore, the Markov property of the chain $(\Pi_k)_{k \geq 0}$ yields

$$\begin{aligned} \Xi' = & \mathbf{E}_{\Pi_1} \left[\sum_{k=0}^{n-2} \sum_{i=1}^q \mathbb{1}_{\{T_k \leq U_{n-1}^\epsilon\}} \mathbb{1}_{\{R_{n-1,k}^\epsilon < t_i^*\}} g \circ \Phi(x_i, R_{n-1,k}^\epsilon) e^{-\Lambda(x_i, R_{n-1,k}^\epsilon)} \Pi_k^i \right. \\ & \left. + \sum_{i=1}^q \mathbb{1}_{\{T_{n-1} \leq U_{n-1}^\epsilon\}} g(x_i) \Pi_{n-1}^i \right]. \end{aligned}$$

In other words, define $w'(\pi) = \mathbf{E} \left[g(X_{U_{n-1}^\epsilon}) \middle| \Pi_0 = \pi \right]$. Using Lemma 4.14, we recognize that $\Xi' = w'(\Pi_1)$. Moreover, thanks to the induction assumption, one has $w'(\pi) \geq v_{N-(n-1)}(\pi) - \epsilon$ so that one obtains

$$\Xi' \geq v_{N-(n-1)}(\Pi_1) - \epsilon. \quad (11)$$

Finally, combining Eq. (10) and (11) and noticing that, according to Lemma 4.7, $\{T_1 \leq U_n^{2\epsilon}\} = \{S_1 \leq r_{n-1}^\epsilon\}$, one obtains

$$\begin{aligned} \mathbf{E}[g(X_{U_n^{2\epsilon}})|\Pi_0 = \pi] &\geq Hg(\pi, r_{n-1}^\epsilon) + \mathbf{E}[v_{N-(n-1)}(\Pi_1) \mathbb{1}_{\{S_1 \leq r_{n-1}^\epsilon\}}|\Pi_0 = \pi] - \epsilon \\ &= J(v_{N-(n-1)}, g)(\pi, r_{n-1}^\epsilon) - \epsilon \\ &\geq v_{N-n}(\pi) - 2\epsilon, \end{aligned}$$

from the definition of r_{n-1}^ϵ . Hence, the result. \square

Theorems 4.26 and 4.29 establish that v_n is the value function of the problem with horizon T_{N-n} and in particular that v_0 is the value function of problem (3).

Introduce now the sequence $(V_n)_{0 \leq n \leq N}$ of random variables defined by $V_n = v_n(\Pi_n)$. In other words, one has $V_N = \sum_{i=1}^q g(x_i) \Pi_N^i$ and for $0 \leq n \leq N-1$, the dynamic programming equation yields the recursion

$$\begin{aligned} V_n &= \sup_{u \in [0; t_q^*]} \mathbf{E}[g \circ \Phi(Z_n, u) \mathbb{1}_{\{S_{n+1} > u\}} + V_{n+1} \mathbb{1}_{\{S_{n+1} \leq u\}} | \Pi_n] \\ &= \max_{m \in M} \left\{ \sup_{u \in [t_m^*; t_{m+1}^*[} \mathbf{E}[g \circ \Phi(Z_n, u) \mathbb{1}_{\{S_{n+1} > u\}} + V_{n+1} \mathbb{1}_{\{S_{n+1} \leq u\}} | \Pi_n] \right\} \\ &\quad \vee \mathbf{E}[V_{n+1} | \Pi_n]. \end{aligned}$$

Our next goal is to provide a numerical scheme based on a discretization of this backward recursion to obtain an approximation of the value function V_0 and derive a family of ϵ -optimal stopping times that can be numerically computed in practice.

5 Numerical approximation by quantization

We are now concerned with numerical approximations. Our approach relies on a discretization of the process $(\Pi_n, S_n)_{0 \leq n \leq N}$, which fully determines our dynamic programming equation. We follow the idea introduced in [2, Section 3.1.]. The key point is the Markov property of the process $(\Pi_n, S_n)_{0 \leq n \leq N}$ stated by Proposition 4.1. Therefore, it is possible to discretize this chain by *quantization* as explained below.

5.1 The quantization approach

There exists an extensive literature on quantization methods for random variables and processes. We do not pretend to present here an exhaustive panorama of these methods. However, the interested reader may for instance, consult the following works [9, 7, 8] and references therein. Consider X an \mathbb{R}^r -valued random variable such that $\|X\|_p < \infty$ where $\|X\|_p$ denotes the L^p -nom of X : $\|X\|_p = (\mathbb{E}[|X|^p])^{1/p}$. Let ν be a fixed integer, the optimal L^p -quantization of the random variable X consists in finding the best possible L^p -approximation of X by a random vector \widehat{X} taking at most ν values: $\widehat{X} \in \{x^1, \dots, x^\nu\}$. This procedure consists in the following two steps:

1. Find a finite weighted grid $\Gamma \subset \mathbb{R}^q$ with $\Gamma = \{x^1, \dots, x^\nu\}$.
2. Set $\widehat{X} = \widehat{X}^\Gamma$ where $\widehat{X}^\Gamma = \text{proj}_\Gamma(X)$ with proj_Γ denotes the closest neighbour projection on Γ .

The asymptotic properties of the L^p -quantization are given by the following result, see e.g. [8].

Theorem 5.1. *If $\mathbb{E}[|X|^{p+\eta}] < +\infty$ for some $\eta > 0$ then one has*

$$\lim_{\nu \rightarrow \infty} \nu^{p/r} \min_{|\Gamma| \leq \nu} \|X - \widehat{X}^\Gamma\|_p^p = J_{p,r} \left(\int |h|^{r/(r+p)}(u) du \right)^{1+p/r},$$

where the distribution of X is $P_X(du) = h(u) \lambda_r(du) + \mu$ with $\mu \perp \lambda_r$, $J_{p,r}$ a constant and λ_r the Lebesgue measure in \mathbb{R}^r .

There exists a similar procedure for the optimal quantization of a Markov chain. Our approximation method is based on the quantization of the Markov chain $(\Pi_k, S_k)_{k \leq N}$. Thus, from now on, we will denote, for $0 \leq k \leq N$, $\Theta_k = (\Pi_k, S_k)$. The CLVQ (Competitive Learning Vector Quantization) algorithm [9, Section 3] provides for each time step $0 \leq k \leq N$ a finite grid Γ_k of $\mathcal{M}_1(E_0) \times \mathbb{R}^+$ as well as the transition matrices $(\hat{Q}_k)_{0 \leq k \leq N-1}$ from Γ_k to Γ_{k+1} . Let $p \geq 1$ such that for all $k \leq N$, Π_k and S_k have finite moments at least up to order p and let $proj_{\Gamma_k}$ be the nearest-neighbor projection from $\mathcal{M}_1(E_0) \times \mathbb{R}^+$ onto Γ_k . The quantized process $(\hat{\Theta}_k)_{k \leq N} = (\hat{\Pi}_k, \hat{S}_k)_{k \leq N}$ with value for each k in the finite grid Γ_k of $\mathcal{M}_1(E_0) \times \mathbb{R}^+$ is then defined by

$$(\hat{\Pi}_k, \hat{S}_k) = proj_{\Gamma_k}(\Pi_k, S_k).$$

We will also denote by Γ_k^Π , the projection of Γ_k on $\mathcal{M}_1(E_0)$, and by Γ_k^S , the projection of Γ_k on \mathbb{R}^+ .

Some important remarks must be made concerning the quantization. On the one hand, the optimal quantization has nice convergence properties stated by Theorem 5.1. Indeed, the L^p -quantization error $\|\Theta_k - \hat{\Theta}_k\|_p$ goes to zero when the number of points in the grids goes to infinity. However, on the other hand, the Markov property is not maintained by the algorithm and the quantized process is generally not Markovian. Although the quantized process can be easily transformed into a Markov chain, this chain will not be homogeneous. It must be pointed out that the quantized process $(\hat{\Theta}_k)_{k \in \mathbb{N}}$ depends on the starting point Θ_0 of the process.

In practice, we begin with the computation of the quantization grids, which merely requires to be able to simulate the process. Notice that in our case, what is actually simulated is the sequence of observation $(Y_k, S_k)_{0 \leq k \leq N}$. We are then able to compute the filter $(\Pi_k)_{0 \leq k \leq N}$ thanks to the recursive equation provided by Proposition 3.4. The grids are only computed once and for all and may be stored off-line. Our schemes are then based on the following simple idea: we replace the process by its quantized approximation within the different recursions. The computation is thus carried out in a very simple way since the quantized process has finite state space.

5.2 Approximation of the value function

Our approximation scheme of the sequence $(V_n)_{0 \leq n \leq N}$ follows the same lines as in [5], but once more, the results therein cannot be applied directly as the Markov chain $(\Theta_k)_{k \in \mathbb{N}}$ is not the underlying Markov chain of some PDMP. Our approach decomposes in two steps. The first one will be to discretize the time-continuous maximization of the operator L to obtain a maximization over a finite set. The second step consists in replacing the Markov chain $(\Theta_n)_{n \in \mathbb{N}} = (\Pi_n, S_n)_{n \in \mathbb{N}}$ by its quantized approximation $(\hat{\Theta}_n)_{n \in \mathbb{N}} = (\hat{\Pi}_n, \hat{S}_n)_{n \in \mathbb{N}}$ within the dynamic programming equation. Thus, the conditional expectations will become easily tractable finite sums.

Definition 5.2. *Let $\Delta > 0$ be such that*

$$\Delta < \frac{1}{2} \min \left\{ |t_i^* - t_j^*| \text{ with } 0 \leq i, j, \leq q \text{ such that } t_i^* \neq t_j^* \right\}. \quad (12)$$

For all $m \in M$, let $Gr_m(\Delta)$ be the finite grid on $[t_m^*; t_{m+1}^*]$ defined as follows

$$Gr_m(\Delta) = \{t_m^* + i\Delta, 1 \leq i \leq i_m\} \cup \{t_{m+1}^* - \Delta\},$$

where $i_m = \max\{i \in \mathbb{N} \text{ such that } t_m^* + i\Delta \leq t_{m+1}^* - \Delta\}$. We also denote $Gr(\Delta) = \cup_{m \in M} Gr_m(\Delta)$.

Remark 5.3. Let $m \in M$. Notice that, thanks to Eq. (12), $Gr_m(\Delta)$ is not empty. Moreover, it satisfies two properties that will be crucial in the sequel:

- a. for all $t \in [t_m^*; t_{m+1}^*]$, there exists $u \in Gr_m(\Delta)$ such that $|u - t| \leq \Delta$,
- b. for all $u \in Gr_m(\Delta)$ and $0 < \eta < \Delta$, one has $[u - \eta; u + \eta] \subset]t_m^*; t_{m+1}^*[$.

A discretized maximization operator L^d is then defined as follows.

Definition 5.4. Let $L^d: B(\mathcal{M}_1(E_0)) \times B(E) \rightarrow B(\mathcal{M}_1(E_0))$ be defined for all $\pi \in \mathcal{M}_1(E_0)$ by

$$L^d(v, h)(\pi) \max_{m \in M} \left\{ \max_{u \in Gr_m(\Delta)} \{J^m(v, h)(\pi, u)\} \right\} \vee Kv(\pi).$$

We now proceed to our second step: replacing the Markov chain $(\Theta_n)_{n \in \mathbb{N}} = (\Pi_n, S_n)_{n \in \mathbb{N}}$ by its quantized approximation $(\hat{\Theta}_n)_{n \in \mathbb{N}} = (\hat{\Pi}_n, \hat{S}_n)_{n \in \mathbb{N}}$ within the operators involved in the construction of the value function.

Definition 5.5. We define the quantized operators \hat{H}_n , \hat{G}_n , \hat{K}_n , \hat{J}_n and \hat{L}_n^d for $n \in \{1, \dots, N\}$, $v \in B(\Gamma_n)$, $h \in B(E)$, $\pi \in \Gamma_{n-1}^\Pi$ and $u \geq 0$ as follows

$$\begin{aligned} \hat{H}_n h(\pi, u) &= \sum_{i=1}^q \pi^i \mathbb{1}_{\{u < t_i^*\}} h \circ \Phi(x_i, u) \mathbf{E}[\mathbb{1}_{\{\hat{S}_n > u\}} | \hat{\Pi}_{n-1} = \pi], \\ \hat{G}_n v(\pi, u) &= \mathbf{E}[v(\hat{\Pi}_n) \mathbb{1}_{\{\hat{S}_n \leq u\}} | \hat{\Pi}_{n-1} = \pi], \\ \hat{K}_n v(\pi) &= \mathbf{E}[v(\hat{\Pi}_n) | \hat{\Pi}_{n-1} = \pi], \\ \hat{J}_n(v, h)(\pi, u) &= \hat{H}_n h(\pi, u) + \hat{G}_n v(\pi, u), \\ \hat{L}_n^d(v, h)(\pi) &= \max_{m \in M} \left\{ \max_{u \in Gr_m(\Delta)} \{\hat{J}_n(v, h)(\pi, u)\} \right\} \vee \hat{K}_n v(\pi). \end{aligned}$$

The quantized approximation of the value function naturally follows.

Definition 5.6. For $0 \leq n \leq N$, define the functions \hat{v}_n on Γ_n^Π as follows

$$\begin{cases} \hat{v}_N(\pi) &= \sum_{i=1}^q g(x_i) \pi^i \quad \text{for all } \pi \in \Gamma_N^\Pi, \\ \hat{v}_{n-1}(\pi) &= \hat{L}_n^d(\hat{v}_n, g)(\pi) \quad \text{for all } \pi \in \Gamma_{n-1}^\Pi \text{ and } 1 \leq n \leq N. \end{cases}$$

For $0 \leq n \leq N$, let $\hat{V}_n = \hat{v}_n(\hat{\Pi}_n)$.

We may now state our main result for the numerical approximation.

Theorem 5.7. *Let $\Delta > 0$ so that for all $0 \leq n \leq N - 1$,*

$$\Delta > (2C_\lambda)^{-1/2} \|S_{n+1} - \hat{S}_{n+1}\|_p^{1/2}$$

then, one has the following bound for the approximation error

$$\begin{aligned} \|V_n - \hat{V}_n\|_p \leq & \|V_{n+1} - \hat{V}_{n+1}\|_p + a\Delta + b\|S_{n+1} - \hat{S}_{n+1}\|_p^{\frac{1}{2}} \\ & + c_n\|\Pi_n - \hat{\Pi}_n\|_p + 2[v_{n+1}]\|\Pi_{n+1} - \hat{\Pi}_{n+1}\|_p, \end{aligned}$$

where $a = [g]_2 + 2C_g C_\lambda$, $b = 4C_g(2C_\lambda)^{\frac{1}{2}}$ and $c_n = [v_n] + 4C_g + 2[v_{n+1}]$ with $[v_n]$, $[v_{n+1}]$ defined in Proposition B.7 and $[g]_2$ defined in Assumption 2.7.

Theorem 5.7 establishes the convergence of our approximation scheme and provides a bound for the rate of convergence. More precisely, it gives a rate for the L^p convergence of \hat{V}_0 towards V_0 . Indeed, one has $\|V_N - \hat{V}_N\|_p = \|\sum_{i=1}^q g(x_i)(\Pi_N^i - \hat{\Pi}_N^i)\|_p \leq C_g\|\Pi_N - \hat{\Pi}_N\|_p$ so that $|V_0 - \hat{V}_0|$ can be made arbitrarily small when the quantization errors $(\|\Theta_n - \hat{\Theta}_n\|_p)_{0 \leq n \leq N}$ go to zero i.e. when the number of points in the quantization grids goes to infinity. In order to prove Theorem 5.7, we proceed similarly to [5] and split the approximation error into four terms $\|V_n - \hat{V}_n\|_p \leq \Xi_1 + \Xi_2 + \Xi_3 + \Xi_4$, with

$$\begin{aligned} \Xi_1 &= \|v_n(\Pi_n) - v_n(\hat{\Pi}_n)\|_p, \\ \Xi_2 &= \|L(v_{n+1}, g)(\hat{\Pi}_n) - L^d(v_{n+1}, g)(\hat{\Pi}_n)\|_p, \\ \Xi_3 &= \|L^d(v_{n+1}, g)(\hat{\Pi}_n) - \hat{L}_{n+1}^d(v_{n+1}, g)(\hat{\Pi}_n)\|_p, \\ \Xi_4 &= \|\hat{L}_{n+1}^d(v_{n+1}, g)(\hat{\Pi}_n) - \hat{L}_{n+1}^d(\hat{v}_{n+1}, g)(\hat{\Pi}_n)\|_p. \end{aligned}$$

The bound for the first term is straightforward from Proposition B.7.

Lemma 5.8. *The first term Ξ_1 is bounded as follows*

$$\|v_n(\Pi_n) - v_n(\hat{\Pi}_n)\|_p \leq [v_n]\|\Pi_n - \hat{\Pi}_n\|_p.$$

The other error terms are studied separately in the following sections.

5.2.1 Second term of the error

For the second error term, we investigate the consequences of replacing the continuous maximization in operator L by a discrete one on $Gr(\Delta)$.

Lemma 5.9. *For all $m \in M$, $v \in B(\mathcal{M}_1(E_0))$ and $\pi \in \mathcal{M}_1(E_0)$ one has*

$$\left| \sup_{u \in [t_m^*, t_{m+1}^*]} J^m(v, g)(\pi, u) - \max_{u \in Gr_m(\Delta)} J^m(v, g)(\pi, u) \right| \leq ([g]_2 + C_g C_\lambda + C_v C_\lambda) \Delta.$$

Proof The function $u \rightarrow J^m(v, h)(\pi, u)$ being continuous, there exists $\bar{t} \in [t_m^*; t_{m+1}^*]$ such that $\sup_{u \in [t_m^*; t_{m+1}^*]} J^m(v, h)(\pi, u) = J^m(v, h)(\pi, \bar{t})$. Moreover, from Remark 5.3.a, one may chose $\bar{u} \in Gr_m(\Delta)$ so that $|\bar{u} - \bar{t}| \leq \Delta$. Propositions B.1 and B.4 stating the Lipschitz continuity of J^m then yield

$$\begin{aligned} 0 &\leq \sup_{u \in [t_m^*; t_{m+1}^*]} J^m(v, h)(\pi, u) - \max_{u \in Gr_m(\Delta)} J^m(v, h)(\pi, u) \\ &\leq J^m(v, h)(\pi, \bar{t}) - J^m(v, h)(\pi, \bar{u}) \\ &\leq ([g]_2 + C_g C_\lambda + C_v C_\lambda) |\bar{t} - \bar{u}| \leq ([g]_2 + C_g C_\lambda + C_v C_\lambda) \Delta. \end{aligned}$$

Hence, the result. \square

Lemma 5.10. *The second term Ξ_2 is bounded as follows*

$$\|L(v_{n+1}, g)(\hat{\Pi}_n) - L^d(v_{n+1}, g)(\hat{\Pi}_n)\|_p \leq ([g]_2 + 2C_g C_\lambda) \Delta.$$

Proof This is a straightforward consequence of the previous lemma once it has been noticed that for all $a, b, c, d \in \mathbb{R}$, one has $|a \vee b - c \vee d| \leq |a - c| \vee |b - d|$. Notice also that Proposition B.7 provides $C_{v_{n+1}} \leq C_g$. \square

5.2.2 Third term of the error

To investigate the third error term, we use the properties of quantization to bound the error made by replacing an operator by its quantized approximation. As in [5], we must first deal with non-continuous indicator functions.

Lemma 5.11. *For all $0 \leq n \leq N - 1$, $m \in M$ and $0 < \eta < \Delta$, one has*

$$\left\| \max_{u \in Gr_m(\Delta)} \mathbf{E}[|\mathbb{1}_{\{S_{n+1} \leq u\}} - \mathbb{1}_{\{\hat{S}_{n+1} \leq u\}}| \hat{\Pi}_n] \right\|_p \leq \eta^{-1} \|S_{n+1} - \hat{S}_{n+1}\|_p + 2\eta C_\lambda.$$

Proof Let $0 < \eta < \Delta$. The difference of the indicator functions equals 1 if and only if S_{n+1} and \hat{S}_{n+1} on either side of u . Therefore, if the difference of the indicator functions equals 1, either $|S_{n+1} - u| \leq \eta$, or $|S_{n+1} - u| > \eta$ and in the latter case $|S_{n+1} - \hat{S}_{n+1}| > \eta$ too since $|S_{n+1} - \hat{S}_{n+1}| > |S_{n+1} - u|$. One has $|\mathbb{1}_{\{S_{n+1} \leq u\}} - \mathbb{1}_{\{\hat{S}_{n+1} \leq u\}}| \leq \mathbb{1}_{\{|S_{n+1} - \hat{S}_{n+1}| > \eta\}} + \mathbb{1}_{\{|S_{n+1} - u| \leq \eta\}}$, leading to

$$\begin{aligned} \left\| \max_{u \in Gr_m(\Delta)} \mathbf{E}[|\mathbb{1}_{\{S_{n+1} \leq u\}} - \mathbb{1}_{\{\hat{S}_{n+1} \leq u\}}| \hat{\Pi}_n] \right\|_p \\ \leq \|\mathbb{1}_{\{|S_{n+1} - \hat{S}_{n+1}| > \eta\}}\|_p + \left\| \max_{u \in Gr_m(\Delta)} \mathbf{E}[\mathbb{1}_{\{|S_{n+1} - u| \leq \eta\}} \hat{\Pi}_n] \right\|_p. \end{aligned}$$

On the one hand, Markov inequality yields

$$\|\mathbb{1}_{\{|S_{n+1} - \hat{S}_{n+1}| > \eta\}}\|_p = \mathbf{P}(|S_{n+1} - \hat{S}_{n+1}| > \eta)^{\frac{1}{p}} \leq \|S_{n+1} - \hat{S}_{n+1}\|_p \eta^{-1}.$$

On the other hand, since $u \in Gr_m(\Delta)$, one has (see Remark 5.3.b) $[u - \eta; u + \eta] \subset]t_m^*; t_{m+1}^*[$ thus S_{n+1} has an absolutely continuous distribution on the interval

$[u - \eta; u + \eta]$ since it does not contain any of the t_i^* . Besides, recall that $\widehat{\Theta}_n = \text{proj}_{\Gamma_n}(\Theta_n)$, hence, $\sigma(\widehat{\Pi}_n) \subset \sigma(\widehat{\Theta}_n) \subset \sigma(\Theta_n)$. We also have $\sigma(\Theta_n) \subset \mathfrak{F}_{T_n}^Y \subset \mathfrak{F}_{T_n}$, the law of iterated conditional expectations provides

$$\begin{aligned} \mathbf{E}[\mathbb{1}_{\{|S_{n+1}-u|\leq\eta\}}|\widehat{\Pi}_n] &= \mathbf{E}\left[\mathbf{E}\left[\mathbb{1}_{\{|S_{n+1}-u|\leq\eta\}}|\mathfrak{F}_{T_n}\right]\mathfrak{F}_{T_n}^Y\right]|\widehat{\Pi}_n] \\ &= \mathbf{E}\left[\mathbf{E}\left[\int_{u-\eta}^{u+\eta}\lambda(\Phi(Z_n,s))ds\right|\mathfrak{F}_{T_n}^Y\right]|\widehat{\Pi}_n] \\ &= \mathbf{E}\left[\sum_{i=1}^q\Pi_n^i\int_{u-\eta}^{u+\eta}\lambda(\Phi(x_i,s))ds\right]|\widehat{\Pi}_n]. \end{aligned}$$

Finally, one obtains $\mathbf{E}[\mathbb{1}_{\{|S_{n+1}-u|\leq\eta\}}|\widehat{\Pi}_n] \leq 2\eta C_\lambda$, hence, the result. \square

Lemma 5.12. *For all $0 \leq n \leq N-1$, one has*

$$\begin{aligned} |Kv_{n+1}(\widehat{\Pi}_n) - \widehat{K}_{n+1}v_{n+1}(\widehat{\Pi}_n)| \\ \leq [v_{n+1}]\mathbf{E}[|\Pi_{n+1} - \widehat{\Pi}_{n+1}||\widehat{\Pi}_n] + (2C_g + 2[v_{n+1}])\mathbf{E}[|\Pi_n - \widehat{\Pi}_n||\widehat{\Pi}_n]. \end{aligned}$$

Proof One has

$$\begin{aligned} |Kv_{n+1}(\widehat{\Pi}_n) - \widehat{K}_{n+1}v_{n+1}(\widehat{\Pi}_n)| \\ = |\mathbf{E}[v_{n+1}(\Pi_{n+1})|\Pi_n = \widehat{\Pi}_n] - \mathbf{E}[v_{n+1}(\widehat{\Pi}_{n+1})|\widehat{\Pi}_n]| \\ \leq |\mathbf{E}[v_{n+1}(\Pi_{n+1})|\Pi_n = \widehat{\Pi}_n] - \mathbf{E}[v_{n+1}(\Pi_{n+1})|\widehat{\Pi}_n]| \\ + |\mathbf{E}[v_{n+1}(\Pi_{n+1}) - v_{n+1}(\widehat{\Pi}_{n+1})|\widehat{\Pi}_n]| = (e) + (f) \end{aligned}$$

On the one hand, Proposition B.7 yields $(f) \leq [v_{n+1}]\mathbf{E}[|\Pi_{n+1} - \widehat{\Pi}_{n+1}||\widehat{\Pi}_n]$. On the other hand, one has $(\widehat{\Pi}_n, \widehat{S}_n) = \text{proj}_{\Gamma_n}(\Pi_n, S_n)$ so that $\sigma(\widehat{\Pi}_n) \subset \sigma(\Pi_n, S_n)$, the law of iterated conditional expectations gives

$$\mathbf{E}[v_{n+1}(\Pi_{n+1})|\widehat{\Pi}_n] = \mathbf{E}\left[\mathbf{E}[v_{n+1}(\Pi_{n+1})|\Pi_n, S_n]|\widehat{\Pi}_n\right].$$

Moreover, Proposition 4.1 yields that the conditional distribution of Π_{n+1} w.r.t. (Π_n, S_n) merely depends on Π_n thus one has $\mathbf{E}[v_{n+1}(\Pi_{n+1})|\Pi_n, S_n] = \mathbf{E}[v_{n+1}(\Pi_{n+1})|\Pi_n]$. One has then

$$\begin{aligned} (e) &= |\mathbf{E}\left[\mathbf{E}[v_{n+1}(\Pi_{n+1})|\Pi_n = \widehat{\Pi}_n] - \mathbf{E}[v_{n+1}(\Pi_{n+1})|\Pi_n]\right]|\widehat{\Pi}_n| \\ &= |\mathbf{E}[Kv_{n+1}(\widehat{\Pi}_n) - Kv_{n+1}(\Pi_n)|\widehat{\Pi}_n]|. \end{aligned}$$

We conclude thanks to Proposition B.5 stating the Lipschitz continuity of operator K and Proposition B.7 concerning the properties of the value function and stating in particular that $C_{v_{n+1}} \leq C_g$. \square

Lemma 5.13. *For all $0 < \eta < \Delta$, an upper bound for the third term Ξ_3 is*

$$\begin{aligned} \|L^d(v_{n+1}, g)(\widehat{\Pi}_n) - \widehat{L}_{n+1}^d(v_{n+1}, g)(\widehat{\Pi}_n)\|_p \\ \leq [v_{n+1}]\|\Pi_{n+1} - \widehat{\Pi}_{n+1}\|_p + (4C_g + 2[v_{n+1}])\|\Pi_n - \widehat{\Pi}_n\|_p \\ + 2C_g(\|S_{n+1} - \widehat{S}_{n+1}\|_p\eta^{-1} + 2\eta C_\lambda). \end{aligned}$$

Proof One has

$$\begin{aligned}
& |L^d(v_{n+1}, g)(\widehat{\Pi}_n) - \widehat{L}_{n+1}^d(v_{n+1}, g)(\widehat{\Pi}_n)| \\
& \leq \max_{m \in M} \left\{ \max_{u \in Gr_m(\Delta)} |J^m(v_{n+1}, g)(\widehat{\Pi}_n, u) - \widehat{J}_{n+1}(v_{n+1}, g)(\widehat{\Pi}_n, u)| \right\} \\
& \quad \vee |Kv_{n+1}(\widehat{\Pi}_n) - \widehat{K}_{n+1}v_{n+1}(\widehat{\Pi}_n)|.
\end{aligned}$$

The term involving operator K was studied in the previous lemma. Let u be a fixed element of $Gr_m(\Delta)$ and define $\alpha(\pi, \pi', s') = \sum_{i=1}^q \pi^i g(\Phi(x_i, u)) \mathbb{1}_{\{s' > u\}} + v_{n+1}(\pi') \mathbb{1}_{\{s' \leq u\}}$. One has then

$$\begin{aligned}
& |J^m(v_{n+1}, g)(\widehat{\Pi}_n, u) - \widehat{J}_{n+1}(v_{n+1}, g)(\widehat{\Pi}_n, u)| \\
& = \left| \mathbf{E}[\alpha(\Pi_n, \Pi_{n+1}, S_{n+1}) | \Pi_n = \widehat{\Pi}_n] - \mathbf{E}[\alpha(\widehat{\Pi}_n, \widehat{\Pi}_{n+1}, \widehat{S}_{n+1}) | \widehat{\Pi}_n] \right| \leq A + B,
\end{aligned}$$

where, proceeding as in Lemma 5.12,

$$\begin{aligned}
A &= |\mathbf{E}[\alpha(\Pi_n, \Pi_{n+1}, S_{n+1}) - \alpha(\widehat{\Pi}_n, \widehat{\Pi}_{n+1}, \widehat{S}_{n+1}) | \widehat{\Pi}_n]|, \\
B &= \left| \mathbf{E}[\mathbf{E}[\alpha(\Pi_n, \Pi_{n+1}, S_{n+1}) | \Pi_n = \widehat{\Pi}_n] - \mathbf{E}[\alpha(\Pi_n, \Pi_{n+1}, S_{n+1}) | \Pi_n] | \widehat{\Pi}_n] \right|.
\end{aligned}$$

Proposition B.7 states that $C_{v_{n+1}} \leq C_g$, thus one has

$$\begin{aligned}
A &\leq C_g \mathbf{E}[|\Pi_n - \widehat{\Pi}_n| | \widehat{\Pi}_n] + [v_{n+1}] \mathbf{E}[|\Pi_{n+1} - \widehat{\Pi}_{n+1}| | \widehat{\Pi}_n] \\
&\quad + 2C_g \mathbf{E}[\mathbb{1}_{\{S_{n+1} \leq u\}} - \mathbb{1}_{\{\widehat{S}_{n+1} \leq u\}} | \widehat{\Pi}_n].
\end{aligned} \tag{13}$$

In the term B , we recognize the operator J^m , $B = \mathbf{E}[J^m(v_{n+1}, g)(\widehat{\Pi}_n, u) - J^m(v_{n+1}, g)(\Pi_n, u) | \widehat{\Pi}_n]$, and from Propositions B.1 and B.4, one has

$$B \leq (3C_g + 2[v_{n+1}]) \mathbf{E}[|\widehat{\Pi}_n - \Pi_n| | \widehat{\Pi}_n]. \tag{14}$$

We gather the bounds provided by Eq. (13), (14) and Lemma 5.12 to obtain

$$\begin{aligned}
& |L^d(v_{n+1}, g)(\widehat{\Pi}_n) - \widehat{L}_{n+1}^d(v_{n+1}, g)(\widehat{\Pi}_n)| \\
& \leq [v_{n+1}] \mathbf{E}[|\Pi_{n+1} - \widehat{\Pi}_{n+1}| | \widehat{\Pi}_n] \\
& \quad + \left((4C_g + 2[v_{n+1}]) \vee (2C_g + 2[v_{n+1}]) \right) \mathbf{E}[|\Pi_n - \widehat{\Pi}_n| | \widehat{\Pi}_n] \\
& \quad + 2C_g \max_{u \in Gr_m(\Delta)} \mathbf{E}[\mathbb{1}_{\{S_{n+1} \leq u\}} - \mathbb{1}_{\{\widehat{S}_{n+1} \leq u\}} | \widehat{\Pi}_n].
\end{aligned}$$

We conclude by taking the L^p norm in the equation above and using Lemma 5.11 to bound the last term. \square

5.2.3 Fourth term of the error

Finally, the fourth error term is bounded using Lipschitz properties.

Lemma 5.14. *The fourth term Ξ_4 is bounded as follows*

$$\begin{aligned}
& \|\widehat{L}_{n+1}^d(v_{n+1}, g)(\widehat{\Pi}_n) - \widehat{L}_{n+1}^d(\widehat{v}_{n+1}, g)(\widehat{\Pi}_n)\|_p \\
& \leq [v_{n+1}] \|\Pi_{n+1} - \widehat{\Pi}_{n+1}\|_p + \|V_{n+1} - \widehat{V}_{n+1}\|_p.
\end{aligned}$$

Proof One has

$$\begin{aligned}
& \|\widehat{L}_{n+1}^d(v_{n+1}, g)(\widehat{\Pi}_n) - \widehat{L}_{n+1}^d(\widehat{v}_{n+1}, g)(\widehat{\Pi}_n)\|_p \\
&= \left\| \max_{m \in M} \max_{u \in Gr_m(\Delta)} \left\{ \widehat{H}_{n+1}g(\widehat{\Pi}_n, u) + \widehat{G}_{n+1}v_{n+1}(\widehat{\Pi}_n, u) \right\} \vee \widehat{K}_{n+1}v_{n+1}(\widehat{\Pi}_n) \right. \\
&\quad \left. - \max_{m \in M} \max_{u \in Gr_m(\Delta)} \left\{ \widehat{H}_{n+1}g(\widehat{\Pi}_n, u) + \widehat{G}_{n+1}\widehat{v}_{n+1}(\widehat{\Pi}_n, u) \right\} \vee \widehat{K}_{n+1}\widehat{v}_{n+1}(\widehat{\Pi}_n) \right\|_p, \\
&\leq \left\| \max_{m \in M} \max_{u \in Gr_m(\Delta)} \mathbf{E} \left[\left(v_{n+1}(\widehat{\Pi}_{n+1}) - \widehat{v}_{n+1}(\widehat{\Pi}_{n+1}) \right) \mathbb{1}_{\{\widehat{S}_{n+1} \leq u\}} \middle| \widehat{\Pi}_n \right] \right. \\
&\quad \left. \vee \mathbf{E}[v_{n+1}(\widehat{\Pi}_{n+1}) - \widehat{v}_{n+1}(\widehat{\Pi}_{n+1}) | \widehat{\Pi}_n] \right\|_p \\
&\leq \|v_{n+1}(\widehat{\Pi}_{n+1}) - \widehat{v}_{n+1}(\widehat{\Pi}_{n+1})\|_p.
\end{aligned}$$

We now introduce $v_{n+1}(\Pi_{n+1})$ to split this term into two differences. The Lipschitz continuity of v_{n+1} stated by Proposition B.7 allows us to bound the first term while we recognize V_{n+1} and \widehat{V}_{n+1} in the second one.

$$\begin{aligned}
& \|\widehat{L}_{n+1}^d(v_{n+1}, g)(\widehat{\Pi}_n) - \widehat{L}_{n+1}^d(\widehat{v}_{n+1}, g)(\widehat{\Pi}_n)\|_p \\
&\leq \|v_{n+1}(\widehat{\Pi}_{n+1}) - v_{n+1}(\Pi_{n+1})\|_p + \|v_{n+1}(\Pi_{n+1}) - \widehat{v}_{n+1}(\widehat{\Pi}_{n+1})\|_p \\
&\leq [v_{n+1}] \|\Pi_{n+1} - \widehat{\Pi}_{n+1}\|_p + \|V_{n+1} - \widehat{V}_{n+1}\|_p.
\end{aligned}$$

Hence, the result. \square

5.3 Numerical construction of an ϵ -optimal stopping time

As in the previous section, we follow the idea of [5] and we use both the Markov chain $(\Theta_n)_{0 \leq n \leq N}$ and its quantized approximation $(\widehat{\Theta}_n)_{0 \leq n \leq N}$ to approximate the expression of the ϵ -optimal stopping time introduced in Definition 4.27. We check that we thus obtain actual stopping times for the observed filtration $(\mathfrak{F}_t^Y)_{t \geq 0}$ and that the expected reward when stopping then is a good approximation of the value function V_0 . For all $(\pi, s) \in \mathcal{M}_1(E_0) \times \mathbb{R}^+$ and $0 \leq n \leq N$, we denote $(\widehat{\pi}_n, \widehat{s}_n) = \text{proj}_{\Gamma_n}(\pi, s)$. Let

$$\widehat{s}_{N-n}^*(\pi, s) = \min\{t \in Gr(\Delta) : \widehat{J}_n(\widehat{v}_n, g)(\widehat{\pi}_{n-1}, t) = \max_{u \in Gr(\Delta)} \widehat{J}_n(\widehat{v}_n, g)(\widehat{\pi}_{n-1}, u)\}.$$

For $1 \leq n \leq N$ and $\pi \in \mathcal{M}_1(E_0)$, we define

$$\widehat{r}_{N-n}(\pi, s) = \begin{cases} t_q^* & \text{if } \widehat{K}_n \widehat{v}_n(\widehat{\pi}_{n-1}) > \max_{u \in Gr(\Delta)} \widehat{J}_n(\widehat{v}_n, g)(\widehat{\pi}_{n-1}, u), \\ \widehat{s}_{N-n}^*(\pi, s) & \text{otherwise.} \end{cases}$$

Let now for $n \geq 1$,

$$\begin{cases} \widehat{R}_{n,0} &= \widehat{r}_{n-1}(\Pi_0, S_0), \\ \widehat{R}_{n,k} &= \widehat{r}_{n-1-k}(\Pi_k, S_k) \mathbb{1}_{\{\widehat{R}_{n,k-1} \geq S_k\}} \text{ for } 1 \leq k \leq n-2, \end{cases}$$

and set $\widehat{U}_n = \sum_{k=1}^n \widehat{R}_{n,k-1} \wedge S_k$. The following result is a direct consequence of Proposition 4.12. It is a very strong result as it states that the numerically computable random variables \widehat{U}_n are actual $(\mathfrak{F}_t^Y)_{t \geq 0}$ -stopping times.

Theorem 5.15. For $0 \leq n \leq N$, \hat{U}_n is an $(\mathfrak{F}_t^Y)_{t \geq 0}$ -stopping time.

We now intend to prove that stopping at time \hat{U}_N provides a good approximation of the value function V_0 . For all $\pi \in \mathcal{M}_1(E_0)$ and $0 \leq n \leq N$ we therefore introduce the expected reward functions when abiding by the stopping rule $(\hat{U}_n)_{0 \leq n \leq N}$ and the corresponding random variables

$$\bar{v}_n(\pi) = \mathbf{E}[g(X_{\hat{U}_{N-n}})|\Pi_0 = \pi], \quad \bar{V}_n = \bar{v}_n(\Pi_n).$$

Theorem 5.16. Let $\Delta > 0$ so that for all $0 \leq n \leq N - 1$,

$$\Delta > (2C_\lambda)^{-1/2} \|S_{n+1} - \hat{S}_{n+1}\|_p^{1/2} 2C_\lambda,$$

one has then the following bound for the error between the expected reward when stopping at time \hat{U}_n and the value function

$$\begin{aligned} \|V_n - \bar{V}_n\|_p \leq & \|V_{n+1} - \bar{V}_{n+1}\|_p + \|V_n - \hat{V}_n\|_p + \|V_{n+1} - \hat{V}_{n+1}\|_p \\ & + d_n \|\Pi_n - \hat{\Pi}_n\|_p + 2[v_{n+1}] \|\Pi_{n+1} - \hat{\Pi}_{n+1}\|_p \\ & + b \|S_{n+1} - \hat{S}_{n+1}\|_p^{1/2}, \end{aligned}$$

where $b = 4C_g(2C_\lambda)^{1/2}$, $d_n = 6C_g + 4[v_{n+1}]$, $[v_{n+1}]$ defined in Proposition B.7.

It is important to notice that $\bar{v}_N(\pi) = \sum_{i=1}^q g(x_i) \pi^i = v_N(\pi)$ and thus $\bar{V}_N = V_N$. Therefore, the previous theorem proves that $|V_0 - \bar{V}_0|$ goes to zero when the quantization errors $(\|\Theta_n - \hat{\Theta}_n\|_p)_{0 \leq n \leq N}$ go to zero. In other words, the expected reward \bar{V}_0 when stopping at the random time \hat{U}_N can be made arbitrarily close to the value function V_0 of the partially observed optimal stopping problem (3) and hence \hat{U}_N is an ϵ -optimal stopping time.

Proof The first step consists in finding a recursion satisfied by the sequence $(\bar{V}_n)_{0 \leq n \leq N}$ in order to compare it with the dynamic programming equation giving $(\hat{V}_n)_{0 \leq n \leq N}$. Let $0 \leq n \leq N - 1$. First of all, Lemma 4.14 gives

$$\begin{aligned} & \mathbf{E}[g(X_{\hat{U}_{N-n}})|\Pi_0] \\ = & \sum_{k=0}^{N-n-1} \sum_{i=1}^q \mathbf{E}[\mathbb{1}_{\{T_k \leq \hat{U}_{N-n}\}} \mathbb{1}_{\{\hat{R}_{N-n,k} < t_i^*\}} g \circ \Phi(x_i, \hat{R}_{N-n,k}) e^{-\Lambda(x_i, \hat{R}_{N-n,k})} \Pi_k^i | \Pi_0] \\ & + \sum_{i=1}^q \mathbf{E}[\mathbb{1}_{\{T_{N-n} \leq \hat{U}_{N-n}\}} g(x_i) \Pi_n^i | \Pi_0]. \end{aligned}$$

The term corresponding to $k = 0$ in the above sum equals $Hg(\Pi_0, \hat{R}_{N-n,0})$. Taking the conditional expectation w.r.t. $\mathfrak{F}_{T_1}^Y$ in the other terms and noticing that one has $\{T_1 \leq \hat{U}_{N-n}\} = \{S_1 \leq \hat{R}_{N-n,0}\}$ yield

$$\mathbf{E}[g(X_{\hat{U}_{N-n}})|\Pi_0] = Hg(\Pi_0, \hat{R}_{N-n,0}) + \mathbf{E}[\Xi'' \mathbb{1}_{\{S_1 \leq \hat{R}_{N-n,0}\}} | \Pi_0],$$

with

$$\begin{aligned} \Xi'' &= \mathbf{E} \left[\sum_{k=1}^{N-n-1} \sum_{i=1}^q \mathbb{1}_{\{T_k \leq \widehat{U}_{N-n}\}} \mathbb{1}_{\{\widehat{R}_{N-n,k} < t_i^*\}} g \circ \Phi(x_i, \widehat{R}_{N-n,k}) e^{-\Lambda(x_i, \widehat{R}_{N-n,k})} \Pi_k^i \right. \\ &\quad \left. + \sum_{i=1}^q \mathbb{1}_{\{T_{N-n} \leq \widehat{U}_{N-n}\}} g(x_i) \Pi_n^i \middle| \mathfrak{F}_{T_1}^Y \right]. \end{aligned}$$

We now intend to apply the Markov property of the sequence $(\Pi_n)_{n \in \mathbb{N}}$ in the term Ξ'' . Similarly to Lemma 4.28, for $n \geq 1$, on the set $\{T_1 \leq \widehat{U}_{N-n}\}$, one has $\widehat{R}_{N-n-1,k-1} \circ \theta = \widehat{R}_{N-n,k}$ for all $1 \leq k \leq n-1$. Thus, on the set $\{T_1 \leq \widehat{U}_{N-n}\}$, one has $\widehat{U}_{N-n} = T_1 + \widehat{U}_{N-n-1} \circ \theta$. Recall that $\mathbb{1}_{\{T_k \leq \widehat{U}_{N-n}\}} = \mathbb{1}_{\{T_{k-1} \leq \widehat{U}_{N-n-1}\}} \circ \theta$. We may therefore apply the Markov property. Using Lemma 4.14, we now obtain $\Xi'' = \bar{v}_{n+1}(\Pi_1)$. Finally, we have

$$\bar{v}_n(\Pi_0) = Hg(\Pi_0, \widehat{R}_{N-n,0}) + G\bar{v}_{n+1}(\Pi_0, \widehat{R}_{N-n,0}) = J(\bar{v}_{n+1}, g)(\Pi_0, \widehat{R}_{N-n,0}).$$

Recall that $\widehat{R}_{N-n,0} = \widehat{r}_{N-n-1}(\Pi_0, S_0)$ and apply the translation operator θ^n to obtain the following recursion

$$\bar{V}_n = J(\bar{v}_{n+1}, g)(\Pi_n, \widehat{r}_{N-n-1}(\Pi_n, S_n)).$$

We are now able to study the error between \bar{V}_n and \widehat{V}_n . Let us recall that, from its definition, $\widehat{r}_{N-n-1}(\Pi_n, S_n)$ equals either $\widehat{s}_{N-n-1}^*(\Pi_n, S_n)$ or t_q^* . In the latter case, notice that $J(\bar{v}_{n+1}, g)(\Pi_n, t_q^*) = K\bar{v}_{n+1}(\Pi_n)$. Eventually, one has

$$|\bar{V}_n - \widehat{V}_n| \leq \mathbb{1}_{\{\widehat{r}_{N-n-1}(\Pi_n, S_n) = t_q^*\}} A + \mathbb{1}_{\{\widehat{r}_{N-n-1}(\Pi_n, S_n) = \widehat{s}_{N-n-1}^*(\Pi_n, S_n)\}} B \leq A \vee B,$$

with

$$\begin{cases} A &= |K\bar{v}_{n+1}(\Pi_n) - \widehat{K}_{n+1}\widehat{v}_{n+1}(\widehat{\Pi}_n)|, \\ B &= |J(\bar{v}_{n+1}, g)(\Pi_n, \widehat{s}_{N-n-1}^*(\Pi_n)) - \max_{u \in Gr(\Delta)} \widehat{J}_{n+1}(\widehat{v}_{n+1}, g)(\widehat{\Pi}_n, u)|. \end{cases}$$

To bound the first term A , we introduce the function v_{n+1} . One has

$$\begin{aligned} A &\leq |K\bar{v}_{n+1}(\Pi_n) - Kv_{n+1}(\Pi_n)| + |Kv_{n+1}(\Pi_n) - Kv_{n+1}(\widehat{\Pi}_n)| \\ &\quad + |Kv_{n+1}(\widehat{\Pi}_n) - \widehat{K}_{n+1}v_{n+1}(\widehat{\Pi}_n)| + |\widehat{K}_{n+1}v_{n+1}(\widehat{\Pi}_n) - \widehat{K}_{n+1}\widehat{v}_{n+1}(\widehat{\Pi}_n)| \\ &\leq (g) + (h) + (i) + (j). \end{aligned}$$

In the above sum, the term (g) is bounded by $\mathbf{E} [|\bar{V}_{n+1} - V_{n+1}| | \Pi_n]$. For the term (h), we use Proposition B.5 stating the Lipschitz continuity of the operator K . The term (i) is bounded by Lemma 5.12 and the term (j) is bounded in the proof of Lemma 5.14. We now turn to the second term B . In the following computations, denote $\widehat{s}^* = \widehat{s}_{N-n-1}^*(\Pi_n, S_n)$. Its definition yields $B = |J(\bar{v}_{n+1}, g)(\Pi_n, \widehat{s}^*) - \widehat{J}_{n+1}(\widehat{v}_{n+1}, g)(\widehat{\Pi}_n, \widehat{s}^*)|$. Once again, we introduce the function v_{n+1} and since one has

$$|J(\bar{v}_{n+1}, g)(\Pi_n, \widehat{s}^*) - J(v_{n+1}, g)(\Pi_n, \widehat{s}^*)| \leq \mathbf{E} [|\bar{V}_{n+1} - V_{n+1}| | \Pi_n],$$

we only have to bound $|J(v_{n+1}, g)(\Pi_n, \widehat{s}^*) - \widehat{J}_{n+1}(\widehat{v}_{n+1}, g)(\widehat{\Pi}_n, \widehat{s}^*)|$. We proceed as for operator K and introduce the quantities $J(v_{n+1}, g)(\widehat{\Pi}_n, \widehat{s}^*)$ and $\widehat{J}_{n+1}(v_{n+1}, g)(\widehat{\Pi}_n, \widehat{s}^*)$. We obtain three differences that are bounded using Propositions B.1 and B.4 for the first one, and with similar arguments as in Lemmas 5.13 and 5.14 respectively for the second and the third ones. \square

6 Numerical example

We apply our procedure to a simple PDMP similar to the one studied in [5]. Let $E = [0; 1[$. For $x \in E$ and $t \geq 0$, the flow is defined by $\Phi(x, t) = x + vt$ so that $t^*(x) = (1 - x)/v$. We set the jump rate to $\lambda(x) = ax$ for some $a > 0$ and the transition kernel $Q(x, \cdot)$ to the uniform distribution on a finite set $E_0 \subset E$. Thus, the process evolves toward 1 and the closer it gets to 1, the more likely it will jump back to some point of E_0 . A trajectory is represented in Figure 1. The observation

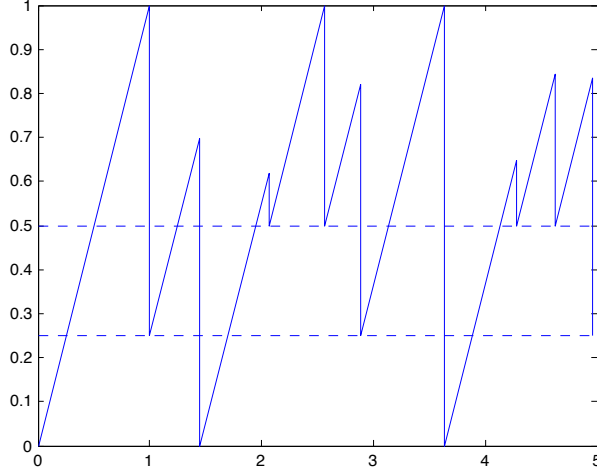


Figure 1: A trajectory of the process drawn until the 9th jump time with $a = 3$, $v = 1$ and $E_0 = \{0; \frac{1}{4}; \frac{1}{2}\}$. The dotted lines represent the possible post-jump values.

process is $Y_n = \varphi(Z_n) + W_n$ where $\varphi(x) = x$ and $W_n \sim \mathcal{N}(0, \sigma^2)$ for some $\sigma^2 > 0$. Finally, we choose the reward function $g(x) = x$. Our assumptions thus clearly hold. Simulations are run with $a = 3$, $v = 1$, $E_0 = \{0; 1/4; 1/2\}$, $\sigma^2 = 0.25$ and $N = 9$. The exact value of V_0 is unknown but one has as in [5],

$$\bar{V}_0 = \mathbf{E}[g(X_{\hat{U}_N})] \leq V_0 = \sup_{\sigma \in \Sigma_N^Y} \mathbf{E}[g(X_\sigma)] \leq \mathbf{E}\left[\sup_{0 \leq t \leq T_N} g(X_t)\right]. \quad (15)$$

Both the first and the last term may be estimated by Monte Carlo simulations. One has thus, with 10^6 trajectories, $\mathbf{E}[\sup_{0 \leq t \leq T_N} g(X_t)] = 0.9944$. The values of \bar{V}_0 , that depend on the quantization grids, are also obtained with 10^6 Monte Carlo simulations and are gathered in Table 1 as well as the approximation \hat{V}_0 of V_0 and the theoretical bound B_{th} of the error $|V_0 - \hat{V}_0|$ provided by Theorem 5.7. This bound decreases as the number of points in the quantization grids increases, as expected. Moreover, Eq. (15) provides an empirical bound $B_{em} = \max\{|\bar{V}_0 - \hat{V}_0|, |\mathbf{E}[\sup_{0 \leq t \leq T_N} g(X_t)] - \hat{V}_0|\}$.

A Computation of a conditional expectation

The objective of this section is to prove the technical Lemma A.2 used in the proof of Lemma 4.14. First, recall some classical result, see e.g. [12].

Quantization grids	Δ	\bar{V}_0	\hat{V}_0	B_{em}	B_{th}
50 points	0.1179	0.7900	0.8135	0.181	683
100 points	0.0970	0.8031	0.8250	0.169	467
300 points	0.0731	0.8182	0.8407	0.154	271
500 points	0.0634	0.8250	0.8477	0.147	211
1000 points	0.0535	0.8313	0.8545	0.140	152
2000 points	0.0453	0.8361	0.8599	0.135	110
4000 points	0.0381	0.8408	0.8643	0.130	80
6000 points	0.0345	0.8430	0.8666	0.128	67
8000 points	0.0321	0.8479	0.8725	0.122	58
10000 points	0.0303	0.8497	0.8742	0.120	53
12000 points	0.0290	0.8521	0.8771	0.117	49

Table 1: Simulation results. The terms B_{em} and B_{th} respectively denote an empirical bound and the theoretical bound provided by Theorem 5.7 for the error $|V_0 - \hat{V}_0|$.

Theorem A.1. *Let X and Y be real-valued integrable random variables defined on a probability space $(\Omega, \mathcal{A}, \mathbf{P})$ with values respectively in two measurable spaces F_1 and F_2 . Let \mathcal{B} be a sub σ -field of \mathcal{A} such that X is \mathcal{B} -measurable and Y is independent from \mathcal{B} . Let $f \in B(F_1 \times F_2)$, one has then*

$$\mathbf{E}[f(X, Y)|\mathcal{B}] = \tilde{f}(X),$$

where $\tilde{f}(x) = \mathbf{E}[f(x, Y)]$.

Lemma A.2. *For all $k \in \mathbb{N}$, one has $\mathbf{E}[\mathbb{1}_{\{S_{k+1} > R_k\}}|\mathfrak{F}_{T_k}] = \mathbb{1}_{\{R_k < t^*(Z_k)\}}e^{-\Lambda(Z_k, R_k)}$.*

Proof First recall some results concerning the random variables $(S_k)_{k \in \mathbb{N}}$, details may be found in [1]. After a jump of the process to the point $z \in E$, the survival function of the time until the next jump is

$$\phi(t, z) = \begin{cases} 1 & \text{if } t \leq 0, \\ e^{-\Lambda(z, t)} & \text{if } 0 \leq t < t^*(z), \\ 0 & \text{if } t \geq t^*(z). \end{cases}$$

Define its generalized inverse $\psi(u, z) = \inf\{t \geq 0 \text{ such that } \phi(t, z) \leq u\}$. Then, for all $k \in \mathbb{N}$, one has $S_{k+1} = \psi(\Upsilon_k, Z_k)$, where Υ_k are i.i.d. random variables with uniform distribution on $[0; 1]$ and independent from \mathfrak{F}_{T_k} . Thus, one has $\mathbf{E}[\mathbb{1}_{\{S_{k+1} > R_k\}}|\mathfrak{F}_{T_k}] = \mathbf{E}[f(\Upsilon_k, Z_k, R_k)|\mathfrak{F}_{T_k}]$ where $f(u, z, r) = \mathbb{1}_{\{\psi(u, z) > r\}}$. As (Z_k, R_k) is \mathfrak{F}_{T_k} -measurable, Υ_k is independent from \mathfrak{F}_{T_k} and $\mathbf{E}[\mathbb{1}_{\{\psi(\Upsilon_k, z) > r\}}] = \mathbb{1}_{\{r < t^*(z)\}}e^{-\Lambda(z, r)}$, Theorem A.1 yields the result. \square

B Lipschitz properties

In this section, we derive the Lipschitz properties of operators H^m , I , G^m , K and L in order to obtain them for the value functions $(v_n)_{0 \leq n \leq N}$.

Proposition B.1. For $m \in M$, $((\pi, u), (\tilde{\pi}, \tilde{u})) \in (\mathcal{M}_1(E_0) \times \mathbb{R}^+)^2$, one has

$$|H^m g(\pi, u) - H^m g(\tilde{\pi}, \tilde{u})| \leq C_g |\pi - \tilde{\pi}| + ([g]_2 + C_g C_\lambda) |u - \tilde{u}|.$$

Proof Since the function $u \rightarrow H^m h(\pi, u)$ is constant on the intervals $[0; t_m^*]$ and $[t_{m+1}^*; +\infty[$, we may assume that $u, \tilde{u} \in [t_m^*; t_{m+1}^*]$ so that one has $H^m g(\pi, u) = \sum_{i=m+1}^q \pi^i e^{-\Lambda(x_i, u)} g \circ \Phi(x_i, u)$, and similarly for $H^m g(\tilde{\pi}, \tilde{u})$. Then, on the one hand, one has

$$\begin{aligned} |H^m g(\pi, u) - H^m g(\tilde{\pi}, u)| &= \left| \sum_{i=m+1}^q (\pi^i - \tilde{\pi}^i) e^{-\Lambda(x_i, u)} g \circ \Phi(x_i, u) \right| \\ &\leq C_g \sum_{i=m+1}^q |\pi^i - \tilde{\pi}^i|. \end{aligned}$$

On the other hand, Lemma A.1 in [5] yields

$$|e^{-\Lambda(x_i, u)} g \circ \Phi(x_i, u) - e^{-\Lambda(x_i, \tilde{u})} g \circ \Phi(x_i, \tilde{u})| \leq ([g]_2 + C_g C_\lambda) |u - \tilde{u}|.$$

Hence, the result. \square

The following technical lemma will be useful to derive the Lipschitz properties of the operator I . The first part of its proof is adapted from [2].

Lemma B.2. For all $\pi, \tilde{\pi} \in \mathcal{M}_1(E_0)$ and $m \in M$, one has

$$\sum_{m=0}^{q-1} \int_{t_m^*}^{t_{m+1}^*} \int_{\mathbb{R}^d} |\Psi(\pi, y', s') - \Psi(\tilde{\pi}, y', s')| \bar{\Psi}_m(\pi, y', s') dy' ds' \leq 2|\pi - \tilde{\pi}|.$$

Proof Let $s' \in [t_m^*; t_{m+1}^*]$ and $y' \in \mathbb{R}^d$. In the following computation, we denote $\tau = (\pi, y', s')$ and $\tilde{\tau} = (\tilde{\pi}, y', s')$, one has

$$\begin{aligned} |\Psi(\tau) - \Psi(\tilde{\tau})| \bar{\Psi}_m(\tau) &= \sum_{j=1}^q \left| \frac{\Psi_m^j(\tau)}{\bar{\Psi}_m(\tau)} - \frac{\Psi_m^j(\tilde{\tau})}{\bar{\Psi}_m(\tilde{\tau})} \right| \bar{\Psi}_m(\tau) \\ &= \sum_{j=1}^q \left| \frac{\Psi_m^j(\tau) \bar{\Psi}_m(\tilde{\tau}) - \Psi_m^j(\tilde{\tau}) \bar{\Psi}_m(\tau)}{\bar{\Psi}_m(\tilde{\tau})} \right| \\ &\leq \sum_{j=1}^q |\Psi_m^j(\tau) - \Psi_m^j(\tilde{\tau})| + \sum_{j=1}^q \frac{\Psi_m^j(\tilde{\tau})}{\bar{\Psi}_m(\tilde{\tau})} |\bar{\Psi}_m(\tau) - \bar{\Psi}_m(\tilde{\tau})|. \end{aligned}$$

Notice that $\sum_{j=1}^q \Psi_m^j(\tilde{\tau}) = \bar{\Psi}_m(\tilde{\tau})$ so that the second sum above reduces to $|\bar{\Psi}_m(\tau) - \bar{\Psi}_m(\tilde{\tau})| = \sum_{j=1}^q |\Psi_m^j(\tau) - \Psi_m^j(\tilde{\tau})|$. Finally, one has

$$|\Psi(\tau) - \Psi(\tilde{\tau})| \bar{\Psi}_m(\tau) \leq 2 \sum_{j=1}^q |\Psi_m^j(\tau) - \Psi_m^j(\tilde{\tau})|.$$

As $\int_{\mathbb{R}^d} f_W(y' - \varphi(x_j)) dy' = 1$ and $\sum_{j=1}^q Q(\Phi(x_i, s'), x_j) = 1$, one obtains

$$\begin{aligned}
& \sum_{m=0}^{q-1} \int_{t_m^*}^{t_{m+1}^*} \int_{\mathbb{R}^d} |\Psi(\pi, y', s') - \Psi(\tilde{\pi}, y', s')| \bar{\Psi}_m(\pi, y', s') dy' ds' \\
& \leq 2 \sum_{m=0}^{q-1} \int_{t_m^*}^{t_{m+1}^*} \sum_{j=1}^q \int_{\mathbb{R}^d} |\Psi_m^j(\pi, y', s') - \Psi_m^j(\tilde{\pi}, y', s')| dy' ds' \\
& \leq 2 \sum_{m=0}^{q-1} \sum_{i=m+1}^q \int_{t_m^*}^{t_{m+1}^*} \sum_{j=1}^q \int_{\mathbb{R}^d} |\pi^i - \tilde{\pi}^i| \lambda(\Phi(x_i, s')) e^{-\Lambda(x_i, s')} \\
& \quad \times Q(\Phi(x_i, s'), x_j) f_W(y' - \varphi(x_j)) dy' ds' \\
& \leq 2 \sum_{m=0}^{q-1} \sum_{i=m+1}^q |\pi^i - \tilde{\pi}^i| \int_{t_m^*}^{t_{m+1}^*} \lambda(\Phi(x_i, s')) e^{-\Lambda(x_i, s')} ds' \\
& \leq 2 \sum_{i=1}^q |\pi^i - \tilde{\pi}^i| \int_0^{t_i^*} \lambda(\Phi(x_i, s')) e^{-\Lambda(x_i, s')} ds'.
\end{aligned}$$

We obtain the result as $\int_0^{t_i^*} \lambda(\Phi(x_i, s')) e^{-\Lambda(x_i, s')} ds' = 1 - e^{-\Lambda(x_i, t_i^*)} \leq 1$. \square

Proposition B.3. For $v \in BL(\mathcal{M}_1(E_0))$ and $((\pi, u), (\tilde{\pi}, \tilde{u})) \in (\mathcal{M}_1(E_0) \times \mathbb{R}^+)^2$, one has

$$|Iv(\pi, u) - Iv(\tilde{\pi}, \tilde{u})| \leq (C_v + 2[v])|\pi - \tilde{\pi}| + C_v C_\lambda |u - \tilde{u}|.$$

Proof On the one hand, one clearly has

$$|Iv(\pi, u) - Iv(\pi, \tilde{u})| \leq \sum_{i=1}^q \pi^i |u \wedge t_i^* - \tilde{u} \wedge t_i^*| C_v C_\lambda \leq C_v C_\lambda |u - \tilde{u}|.$$

On the other hand, one has

$$\begin{aligned}
& |Iv(\pi, u) - Iv(\tilde{\pi}, u)| \\
& \leq C_v |\pi - \tilde{\pi}| + \sum_{i=1}^q \pi^i \int_0^{t_i^*} \int_{\mathbb{R}^d} |v(\Psi(\pi, y', s')) - v(\Psi(\tilde{\pi}, y', s'))| \\
& \quad \times \sum_{j=1}^q Q(\Phi(x_i, s'), x_j) f_W(y' - \varphi(x_j)) \lambda(\Phi(x_i, s')) e^{-\Lambda(x_i, s')} dy' ds'.
\end{aligned}$$

Besides, we have assumed that v is Lipschitz continuous so that one has

$$|v(\Psi(\pi, y', s')) - v(\Psi(\tilde{\pi}, y', s'))| \leq [v] |\Psi(\pi, y', s') - \Psi(\tilde{\pi}, y', s')|.$$

Thus, one has

$$\begin{aligned}
& |Iv(\pi, y, s, u) - Iv(\tilde{\pi}, y, s, u)| \\
& \leq C_v |\pi - \tilde{\pi}| + [v] \sum_{i=1}^q \pi^i \int_0^{t_i^*} \int_{\mathbb{R}^d} \left| \Psi(\pi, y', s') - \Psi(\tilde{\pi}, y', s') \right| \\
& \quad \sum_{j=1}^q Q\left(\Phi(x_i, s'), x_j\right) f_W(y' - \varphi(x_j)) \lambda \circ \Phi(x_i, s') e^{-\Lambda(x_i, s')} dy' ds' \\
& \leq C_v |\pi - \tilde{\pi}| + [v] \sum_{m=0}^{q-1} \sum_{i=m+1}^q \pi^i \int_{t_m^*}^{t_{m+1}^*} \int_{\mathbb{R}^d} \left| \Psi(\pi, y', s') - \Psi(\tilde{\pi}, y', s') \right| \\
& \quad \times \sum_{j=1}^q Q\left(\Phi(x_i, s'), x_j\right) f_W(y' - \varphi(x_j)) \lambda \circ \Phi(x_i, s') e^{-\Lambda(x_i, s')} dy' ds' \\
& \leq C_v |\pi - \tilde{\pi}| + [v] \sum_{m=0}^{q-1} \int_{t_m^*}^{t_{m+1}^*} \int_{\mathbb{R}^d} \left| \Psi(\pi, y', s') - \Psi(\tilde{\pi}, y', s') \right| \bar{\Psi}_m(\pi, y', s') dy' ds'.
\end{aligned}$$

The previous lemma provides the result. \square

Proposition B.4. For $m \in M$, $v \in BL(\mathcal{M}_1(E_0))$ and $((\pi, u), (\tilde{\pi}, \tilde{u})) \in (\mathcal{M}_1(E_0) \times \mathbb{R}^+)^2$, one has

$$|G^m v(\pi, u) - G^m v(\tilde{\pi}, \tilde{u})| \leq (2C_v + 2[v])|\pi - \tilde{\pi}| + C_v C_\lambda |u - \tilde{u}|.$$

Proof As in the proof of Proposition B.1, we may assume without loss of generality that $u, \tilde{u} \in [t_m^*, t_{m+1}^*]$ so that one has

$$\begin{aligned}
G^m v(\pi, u) &= Iv(\pi, u) + \sum_{i=1}^m \pi^i e^{-\Lambda(x_i, t_i^*)} \\
&\quad \times \sum_{j=1}^q \int_{\mathbb{R}^d} v\left(\Psi(\pi, y', t_i^*)\right) Q\left(\Phi(x_i, t_i^*), x_j\right) f_W(y' - \varphi(x_j)) dy',
\end{aligned}$$

and similarly for $G^m v(\tilde{\pi}, \tilde{u})$. The second term does not depend on u thus

$$\begin{aligned}
|G^m v(\pi, u) - G^m v(\pi, \tilde{u})| &= |Iv(\pi, u) - Iv(\pi, \tilde{u})| \\
&\leq |Iv(\pi, u) - Iv(\tilde{\pi}, u)| + C_v |\pi - \tilde{\pi}|,
\end{aligned}$$

as $\Psi(\pi, y', t_i^*) = \Psi(\tilde{\pi}, y', t_i^*)$ by Proposition 3.4. This yields the result. \square

Proposition B.5. For all $v \in BL(\mathcal{M}_1(E_0))$ and $(\pi, \tilde{\pi}) \in \mathcal{M}_1(E_0)^2$, one has

$$|Kv(\pi) - Kv(\tilde{\pi})| \leq (2C_v + 2[v])|\pi - \tilde{\pi}|.$$

Proof The proof is similar to the previous one since $Kv(\pi) = Gv(\pi, t_q^*)$. \square

Proposition B.6. For $v \in BL(\mathcal{M}_1(E_0))$ and $(\pi, \tilde{\pi}) \in \mathcal{M}_1(E_0)^2$, one has

$$|L(v, g)(\pi) - L(v, g)(\tilde{\pi})| \leq (C_g + 2C_v + 2[v])|\pi - \tilde{\pi}|.$$

Proof One has

$$\begin{aligned}
& |L(v, g)(\pi) - L(v, g)(\tilde{\pi})| \\
& \leq \max_{m \in M} \left\{ \sup_{u \in [t_m^*, t_{m+1}^*]} |J^m(v, g)(\pi, u) - J^m(v, g)(\tilde{\pi}, u)| \right\} \vee |Kv(\pi) - Kv(\tilde{\pi})| \\
& \leq (C_g + 2C_v + 2[v]) |\pi - \tilde{\pi}|,
\end{aligned}$$

using Propositions B.1, B.4 and B.5 since $J^r(v, g) = H^r g + G^r v$. \square

Proposition B.7. *For all $n \in \{0, \dots, N\}$, one has $v_n \in BL(\mathcal{M}_1(E_0))$ with $C_{v_n} \leq C_g$ and $[v_n] \leq (2^{N-n+2} - 3)C_g$.*

Proof We proved that v_n is the value function of the optimal stopping problem with horizon T_{N-n} thus one has $v_n(\pi) = \sup_{\sigma \in \Sigma_{N-n}^Y} \mathbf{E}[g(X_\sigma) | \Pi_0 = \pi] \leq C_g$. Therefore v_n is bounded and $C_{v_n} \leq C_g$. The second assessment is proved by backward induction. Let $\pi, \tilde{\pi} \in \mathcal{M}_1(E_0)$. One has

$$|v_N(\pi) - v_N(\tilde{\pi})| \leq \sum_{j=1}^N g(x_j) |\pi^j - \tilde{\pi}^j| \leq C_g |\pi - \tilde{\pi}|.$$

Therefore, we have the result for $n = N$ with $[v_N] \leq C_g$. Moreover, since $v_n = L(v_{n+1}, g)$ for $0 \leq n \leq N - 1$, Proposition B.6 yields $[v_n] \leq 3C_g + 2[v_{n+1}]$ which proves the propagation of the induction. \square

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